A New SLM Algorithm with Low Complexity for PAPR Reduction in OFDM

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Abstract

The peak to average power ratio (PAPR) distribution of the periodic orthogonal frequency division multiplexing (OFDM) data symbol sequence is evaluated and it is shown that the OFDM data symbol sequence with short period is expected to have high PAPR. Using this result, we identify the optimal phase sequence set and propose rows of a cyclic Hadamard matrix constructed by m-sequence as phase sequences for selected mapping (SLM) OFDM. The computational complexity of conventional SLM OFDM is increased extensively in proportion to the number of phase sequences. In order to overcome this disadvantage, we propose new SLM OFDM scheme. From the simulation results for the standard of IEEE 802.16 of mobile wireless metropolitan area network (WMAN), it is shown that new SLM OFDM with 2048 carriers reduces the computational complexity up to 51% while it has almost the same capability of PAPR reduction compared to that of conventional SLM OFDM. Since the computational reduction gain increases as the number of carriers increases, the proposed scheme is appropriate for the high data rate OFDM systems.

Keywords – Orthogonal frequency division multiplexing (OFDM), peak to average power ratio (PAPR), selected mapping (SLM)

I. Introduction

Orthogonal frequency division multiplexing (OFDM) system is one of the strong candidates for the standard of the next generation mobile radio communication systems and it has been accepted as a standard for the wireless local area network (WLAN) and mobile wireless metropolitan area network (WMAN).

One of the major drawbacks of OFDM system is that it has a high PAPR. The high PAPR brings on signal distortion at a high power amplifier (HPA) because of the non-linearity of the HPA. Then, the signal distortion induces the degradation of bit error rate (BER). A lot of techniques ([1]-[5]) have been proposed to reduce PAPR problem of OFDM systems. The methods can be classified into two categories. First, there is a deterministic method that limits PAPR of OFDM below a certain level. Clipping and block coding belong to this category. Clipping is widely used from the simplicity of its implementation but it introduces both in-band and out-of-band radiation. Block coding [5] reduces PAPR by encoding the input data to codeword that has low PAPR but it increases the redundant symbols seriously.

The second category is based on probabilistic approach. This method improves the characteristic of PAPR statistically without signal distortion. SLM and partial transmit sequence (PTS) [1] are classified as this category. In SLM, statistically independent alternative OFDM data symbol sequences are generated by multiplying predetermined U phase sequences with an input OFDM data symbol sequence. The symbol sequence with the lowest PAPR is selected for transmission after IFFT's. In PTS, the input data symbol sequence is partitioned into a number of disjoint sub-blocks. IFFT is applied to each group and the signal of each group is multiplied by a rotating factor. Then, the PAPR is computed for each set of phase rotating combinations and compared. It is known that SLM is more advantageous than PTS if the amount of redundancy is limited but the computational complexity is larger than PTS.

In this paper, new SLM OFDM is proposed to reduce PAPR with low computational complexity and it is shown
that new SLM OFDM reduces the computational complexity while it has almost same capability of PAPR reduction compared to that of conventional SLM OFDM.

II. PAPR Distribution

A. OFDM System Model

The discrete time domain OFDM signal of \( N \) carriers in the time interval of \( 0 \leq t \leq N - 1 \) can be written as

\[
a[t] = \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} A[n] e^{j2\pi \frac{n}{N} t}
\]  

(1)

where \( A[n] \) is the input data symbol of \( n \)-th carrier and discrete time \( t \) is integer value. The PAPR of OFDM signal, defined as the ratio of the maximum power divided by the average power of the signal, is expressed as

\[
PAPR \triangleq \frac{\text{Max}[a[t]]^2}{E[|a[t]|^2]}, \quad 0 \leq t \leq N - 1.
\]

The crest factor \( \zeta \) is defined as

\[
\zeta \triangleq \sqrt{\text{PAPR}}, \quad 0 \leq t \leq N - 1.
\]

The PAPR of continuous time domain signal can be estimated from the PAPR of discrete time domain signal, which is over-sampled with the over-sampling factor not less than 8.

B. PAPR Distribution of Periodic Input Symbol Sequence

Let \( X[n] \) be an input data symbol sequence of length \( N \), which has nonzero complex value only within the interval \( 0 \leq n \leq N/M - 1 \).

\[
X[n] = \begin{cases} 
\text{nonzero}, & 0 \leq n \leq N/M - 1 \\
0, & \text{otherwise}.
\end{cases}
\]

If sequence \( A[n] \) of length \( N \) is generated by repeating the non-zero value of \( X[n] \) \( M \) times, \( A[n] \) is expressed as

\[
A[n] = \sum_{m=0}^{M-1} X \left[ n - \frac{N}{M} m \right].
\]

Let \( a[t] \) and \( x[t] \) denote discrete time domain OFDM signals of \( A[n] \) and \( X[n] \) respectively. According to the shift property of Fourier transformation, \( a[t] \) can be expressed as

\[
a[t] = \sum_{n=0}^{M-1} x[n] e^{j2\pi \frac{n}{M} t}.
\]

(2)

In [1], the PAPR distribution of \( a[t] \) with \( M = 1 \) has been evaluated. In the following, we will derive the PAPR distribution for \( M \geq 1 \). We first calculate the distribution of instantaneous power of \( x[t] \). As \( X[n] \) has nonzero value only within the interval \( 0 \leq n \leq N/M - 1 \), OFDM signal \( x[t] \) is given as

\[
x[t] = \frac{1}{\sqrt{N}} \sum_{n=0}^{N/M-1} X[n] e^{j2\pi \frac{n}{N} t}.
\]

(3)

As \( X[n] \) can be thought as independent zero-mean random variable with variance \( \sigma_x^2 \), the average power \( \sigma_x^2 \) of OFDM signal \( x[t] \) is calculated using Parseval's theorem

\[
\sigma_x^2 \triangleq E[|x[t]|^2] = \frac{N}{M} \sigma_x^2 = \frac{\sigma_x^2}{M}.
\]

(4)

Since \( x[t] \) is the sum of \( N/M \) nonzero values of \( X[n] \), becomes a zero-mean complex near Gaussian distributed random variable with variance \( \sigma_x^2 \) due to the central limit theorem for large \( N/M \). \( r = |x| \) has Rayleigh distribution and is written as

\[
p_r(r) = \frac{2r}{\sigma_x^2} \exp\left(-\frac{r^2}{\sigma_x^2}\right).
\]

(5)

From (2) and (5), nonzero \( u = |a| \) has Rayleigh distribution as follows:

\[
p_a(u) = \frac{2u}{M^2 \sigma_x^2} \exp\left(-\frac{u^2}{M^2 \sigma_x^2}\right).
\]

(6)

From (4) and (6), the probability that the magnitude of one OFDM signal \( a[t] \) value does not exceed \( a_0 \) is given by

\[
\Pr\{|a| \leq a_0\} = \int_0^{a_0} p_a(u) \, du = 1 - \exp\left(-\frac{a_0^2}{M^2 \sigma_x^2}\right)
\]

(7)

In (2), \( a[t] \) is zero for \( t \neq 0 \mod M \). Assuming
to be statistically independent, the probability that
at least one magnitude value of OFDM signal $a[t]$ exceeds $a_0$ can be approximated as

$$\Pr\{\exists t, 0 \leq t \leq N - 1, |a[t]| > a_0\} = 1 - \Pr\{\max_{0 \leq t \leq N - 1} |a[t]| \leq a_0\} = 1 - \left(1 - \exp\left(-\frac{a_0^2}{M \sigma^2_a}\right)\right)^{N/M}. \quad (8)$$

Using Parseval’s theorem, the average power $\sigma^2_a$ of OFDM signal $a[t]$ is calculated as

$$\sigma^2_a = \mathbb{E}[|a[t]|^2] = \sigma^2_a. \quad (9)$$

Plugging (9) into (8), the probability that the crest factor exceeds some threshold value $\zeta_0 = a_0/\sigma_a$ can be written as

$$\Pr\{\zeta > \zeta_0\} = 1 - \left(1 - \exp\left(-\frac{\zeta_0^2}{M}\right)\right)^{N/M}. \quad (10)$$

From (10), the probability that OFDM signal exceeds the value $\Pr_{\text{OFDM}} = \zeta_0^2$ becomes large as $M$ increases.

C. PAPR Bound of Two Sequences

Let $A[n]$ and $B[n]$ be binary input data symbol sequences of length $N$. When the distance of two sequences is $D$, $B[n]$ can be expressed by

$$B[n] = A[n] - 2 \sum_{d=0}^{D-1} A[n] \delta[n-n_d]. \quad (11)$$

From (11), IFFT of $B[n] - A[n]$ can be calculate as

$$\text{IFFT}\{B[n] - A[n]\} = -2 \frac{1}{\sqrt{N}} \sum_{d=0}^{D-1} A[n_d] e^{j2\pi n_d/N}. \quad (12)$$

Let $a[t]$ and $b[t]$ denote OFDM signals of $A[n]$ and $B[n]$ respectively, $|b[t] - a[t]|$ can be written as

$$|b[t] - a[t]| \leq 2 \frac{1}{\sqrt{N}} \sum_{d=0}^{D-1} A[n_d] e^{j2\pi n_d/N} = 2D \frac{1}{\sqrt{N}}. \quad (13)$$

From (13), $|b[t]|$ is bounded as

$$|a[t]| - \frac{2D}{\sqrt{N}} \leq |b[t]| \leq |a[t]| + \frac{2D}{\sqrt{N}}. \quad (14)$$

In case of binary sequence, $a[t]$ and $b[t]$ have the average power 1. Then, the crest factor of $b[t]$ can be written as

$$\zeta_b - \frac{2D}{\sqrt{N}} \leq \zeta_0 \leq \zeta_a + \frac{2D}{\sqrt{N}}. \quad (15)$$

The difference of the crest factor of $a[t]$ and $b[t]$ becomes very small for large number of carriers $N$ and relatively small distance $D$.

III. New SLM OFDM

A. Conventional SLM OFDM

In conventional SLM OFDM [1], $U$ statistically independent alternative OFDM input data symbol sequences are generated by multiplying pre-determined sequence with an input data symbol sequence, where the pre-determined sequence is called a phase sequence.

After $U$ IFFTs for alternative OFDM input data symbol sequences, the signal with the lowest PAPR is selected for transmission. Assume that the alternative input data symbol sequences are statistically independent.

Using (10) with $M = 1$, the probability that the crest factor exceeds the threshold value can be written as

$$\Pr_{\text{SLM OFDM}}\{\zeta > \zeta_0\} = \left(1 - \left(1 - \exp\left(-\zeta_0^2\right)\right)^{U}\right).$$

The computational complexity of conventional SLM OFDM is increased in proportion to the number of phase sequences.

B. New SLM OFDM

New SLM OFDM transforms input data symbols into a set of alternative time domain signals. Then, the time domain signal that has the lowest PAPR is selected for transmission. In this scheme, the $N = 2^k$ points IFFT is partitioned into two parts. The first part is IFFT of the first $k$ stage and the second part is IFFT of the remaining $(n-k)$ stage. Alternative OFDM signals are generated by multiplying different phase sequences with the intermediate result at $k$-th stage of IFFT.
As the intermediate result is used commonly, the only \((n-k)\) stages after the \(k\)-th stage of IFFT are to be calculated \(U\) times for \(U\) phase sequences. In order to recover input data symbol sequence from the received signal, the receiver needs to have information on which phase sequence is used in the transmitter. A simple method for transmitting the information is to add the index of phase sequence into input data symbol sequence. The bits for transmitting the side information are called redundancy bits. As redundancy bits are very important, it should be coded for error detection and error correction. If \(M\)-QAM constellation and \(U\) phase sequences are used, the number of redundancy bits is \(\lceil \log_2 U/r \rceil\) where \(r\) is coding rate. \(\lceil \log_2 U/r \rceil\) carriers are set to zero to reserve side information. To implement this without further increase of computational complexity, the IFFT signals \(a_{\text{index}}^{(n)}\) of the coded index \((1\sim U)\) are stored in the memory. Then, the side information is added after the IFFT of input data symbol sequence. The new SLM OFDM shown in Fig. 1 can be written as

\[
\begin{align*}
a^{(n)} = \ & \text{IFFT}_{k+1} \left[ p^{(n)} \cdot \text{IFFT}_{k} (A_{\text{data}}) \right] + \text{IFFT}_{1} (A_{\text{index}}) \\
= \ & \text{IFFT}_{k+1} \left[ p^{(n)} \cdot a_{\text{data}}^{(k)} \right] + a_{\text{index}}^{(n)},
\end{align*}
\]

where \(\text{IFFT}_{j}^{i}\) indicates IFFT from \(i\) stage to \(j\) stage and the size of IFFT is \(N=2^{n}\).

C. Computational Complexity

The number of complex computation of conventional SLM OFDM when the number of carriers is \(N=2^{n}\), is given by

\[
\begin{align*}
n_{\text{mul}} = 2^{n-1} n U \\
n_{\text{add}} = 2^{n} n U,
\end{align*}
\]

where \(U\) is the number of phase sequences. When the phase sequences are multiplied at the \(k\)-th stage, the number of complex computation of new SLM OFDM is given by

\[
\begin{align*}
n_{\text{mul}} = 2^{n-1} n + 2^{n-1} (n-k) (U-1) \\
n_{\text{add}} = 2^{n} n + 2^{n} (n-k) (U-1).
\end{align*}
\]

Table 1 gives complexity reduction gain of new SLM OFDM against conventional SLM OFDM with various numbers of sequences and multiplication stages.

<table>
<thead>
<tr>
<th>(N)</th>
<th>(U=256) (n=8)</th>
<th>(U=16) (n=8)</th>
<th>(U=64) (n=11)</th>
<th>(U=256) (n=11)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(n)</td>
<td>(U=4)</td>
<td>(U=8)</td>
<td>(U=16)</td>
<td>(U=4)</td>
</tr>
<tr>
<td>3</td>
<td>46.9</td>
<td>54.7</td>
<td>58.6</td>
<td>54.5</td>
</tr>
<tr>
<td>4</td>
<td>37.5</td>
<td>43.8</td>
<td>46.9</td>
<td>47.7</td>
</tr>
<tr>
<td>5</td>
<td>28.1</td>
<td>32.8</td>
<td>35.2</td>
<td>40.9</td>
</tr>
<tr>
<td>6</td>
<td>18.8</td>
<td>21.9</td>
<td>23.4</td>
<td>34.1</td>
</tr>
<tr>
<td>7</td>
<td>9.4</td>
<td>10.9</td>
<td>11.7</td>
<td>27.3</td>
</tr>
</tbody>
</table>

IV. Phase Sequences

A. Phase Sequences of Conventional SLM OFDM

In SLM OFDM, the two important problems on phase sequences are how to design phase sequences to generate \(U\) statistically independent OFDM data symbol and how many phase sequences to use. It is known from simulation results that the random sequences show the
best performance [3], but there has been no clear way to identify the optimal phase sequence set.

It has been used without any mathematical proof that the phase sequences are to be orthogonal for generating statistically independent alternative OFDM data symbol sequences from an input data symbol sequence.

Fig. 2 shows the complementary cumulative distribution function (CCDF) of the PAPR of SLM OFDM system with 64 carriers for some phase sequences where the number $U$ of phase sequences is 4. The performance of binary random sequence, quaternary random sequence, cyclic Hadamard sequences, $Z_4$ family-A sequences is almost the same. But the performance of Sylvester Hadamard sequences is not so good as that of random or pseudo-random sequences though Sylvester Hadamard sequences are orthogonal.

![CCDF of the PAPR of SLM OFDM for some phase sequences.](image)

Fig. 2. CCDF of the PAPR of SLM OFDM for some phase sequences.

From this result, we can think that the orthogonal property between phase sequences is not a sufficient condition for the optimal phase sequences in SLM OFDM. In section II, we have shown that if an input data symbol sequence of OFDM system is periodic or similar to periodic sequence, OFDM signal has the large PAPR with high probability. If a random phase sequence is multiplied with a periodic input data symbol sequence, the obtained sequence may get aperiodic property and has the low PAPR.

A cyclic Hadamard matrix is a Hadamard matrix with a property that in the standard form, removing the top row and the left-most column, the rows are cyclic shifts of each other. The rows of cyclic Hadamard matrix are orthogonal and have good aperiodic autocorrelation property (large merit factor) except the first row. Thus, the rows of cyclic Hadamard matrix are suited to the phase sequences of conventional SLM OFDM.

B. Phase Sequences of New SLM OFDM

In new SLM OFDM, the $U$ phase sequences are multiplied with the intermediate results of IFFT while phase sequences are multiplied with the input data symbol sequences in SLM OFDM. The computational complexity of new SLM OFDM is reduced as the intermediate multiplication stage $k$ approaches to the last stage $n$. But the performance of PAPR reduction decreases. As there is trade-off between the computational complexity and the performance of PAPR reduction, it is very important to compromise them by finding an optimal stage from simulation results.

For any $N = 2^n$, $N$ point IFFT can be computed from two $N/2$ point IFFT, the $N/2$ point IFFT from two $N/4$ point IFFT and so on. As the transform of $k$-th stage consists of $2^{n-k}$ blocks that are $N/2^{n-k}$ point IFFT according to the successive doubling method, we let the elements of phase sequence in one block have the same value. After generating sequence $R^{(v)}$ of the length $2^{n-k}$, phase sequence $P^{(v)}$ of the length $N$ is made by repeating each element of $R^{(v)}$, $N/2^{n-k}$ times.

$P^{(v)}_{1:N/2^{n-k}} = \cdots = P^{(v)}_{N/2^{n-k}+N/2^{n-k}+1} = R^{(v)}_i, \quad 0 \leq i < 2^{n-k}$.

In new SLM OFDM, rows of a cyclic Hadamard matrix are used as sequences $R^{(v)}$.

V. Simulation Results

Simulations are performed for the standard of IEEE 802.16 [6] for mobile WMAN. The OFDM system specified in IEEE 802.16 has 2048 carriers, which are modulated with QPSK, 16QAM, and 64QAM signal constellation according to the transmission rate. The number of used carriers is 1702 and 346 carriers are set to zero to shape the power spectrum density of the transmit signal. The 100,000 input data symbols are generated randomly to provide the statistics of PAPR.

A. Varying the Stage of Multiplication

Fig. 3 shows the simulation results as the multiplication
The PAPR distribution of the periodic OFDM data symbols has been evaluated and it has been shown that the OFDM data symbol sequence with large periodicity is expected to have high PAPR. Then, we have identified the optimal phase sequence set and propose rows of a cyclic Hadamard matrix as a phase sequence for SLM OFDM system. The new SLM OFDM has been proposed and its performance analyzed in reference to the standard of IEEE 802.16. As results of simulations, the new SLM OFDM with 2048 carriers reduces the computational complexity up to 51% while it has almost the same capability of PAPR reduction compared to that of the conventional SLM OFDM. Since the computational reduction gain increases as the number of carriers increase, the proposed scheme is appropriate for the high data rate OFDM systems.

B. Performance Comparison between Conventional SLM OFDM and New SLM OFDM

Fig. 4 shows a comparison of the performance between conventional SLM OFDM and new SLM OFDM when the multiplication stage \((n-k)\) is 5. The new SLM OFDM reduces the computational complexity up to 41% ~ 51% as the number of sequences \(U\) increases from 4 to 16.

VI. Conclusion