Design of the Phase Sequences for Selected Mapping OFDM System

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Abstract

Peak to average power ratio (PAPR) distribution of the periodic orthogonal frequency division multiplexing (OFDM) symbol sequences is evaluated and it is shown that the OFDM symbol sequence with short period is expected to have high PAPR. PAPR bound of two sequences with Hamming distance D is derived. Using these results, we identify two conditions for the optimal phase sequences of selected mapping (SLM) and propose rows of a cyclic Hadamard matrix constructed by m-sequence as near optimal phase sequences for SLM, which has the best performance to reduce the PAPR of OFDM signals with small redundancy and no signal distortion.

I. INTRODUCTION

One of the major drawbacks of orthogonal frequency division multiplexing (OFDM) system is that it has a high PAPR. Due to the non-linearity of a high power amplifier (HPA), the high PAPR causes signal distortion in the nonlinear HPA. The signal distortion induces the degradation of bit error rate (BER).

Selected mapping (SLM) OFDM system [1], [4] statistically improves the characteristic of PAPR for OFDM signal without signal distortion. In SLM OFDM system, alternative OFDM input symbol sequences are generated by multiplying phase sequences with an input symbol sequence. The OFDM signal with the lowest PAPR is selected for transmission after inverse fast Fourier transform (IFFT).

It has been known from simulation results that random sequences show the best performance [3], but there has been no clear way to identify the optimal phase sequence set. In this paper, we identify two conditions for the optimal phase sequences and propose rows of a cyclic Hadamard matrix constructed by m-sequence as good phase sequences for SLM OFDM system.

II. SLM OFDM SYSTEM

The discrete time domain OFDM signal of N-carriers in the time interval, $0 \le t \le N-1$ can be expressed as

$$a[t] = \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} A[n] e^{j2\pi \frac{n}{N} t}$$

where $A[n]$ is an input symbol sequence and t is a discrete time index.

The PAPR and the crest factor of OFDM signal $a[t]$ are expressed as

$$\text{PAPR}_a = \zeta_a^2 \triangleq \frac{Max\left|a[t]\right|^2}{E\left[\left|a[t]\right|^2\right]}, \quad 0 \le t \le N-1. \quad (1)$$

In the SLM OFDM system [1], alternative input symbol sequences $\boldsymbol{A}^{(u)}$, $1 \le u \le U$ are generated by multiplying phase sequences $\boldsymbol{P}^{(u)}$ with an input symbol sequence $\boldsymbol{A}$, i.e., $A^{(u)}[n] = A[n] P^{(u)}[n]$, $0 \le n \le N-1$. Each element of phase sequences has unit magnitude to preserve the same power and the first phase sequence $\boldsymbol{P}^{(1)}$ is all one sequence $\boldsymbol{1}_N$. After U inverse fast Fourier transforms (IFFT) of $\boldsymbol{A}^{(u)}$, the OFDM signal with the lowest PAPR is selected for transmission. In order to find input symbol sequence from the received signal, the receiver needs to have the information on which phase sequence is used in the transmitter. A simple method for transmitting the information is to add the index of phase sequence into input symbol sequence. The block diagram of SLM OFDM system is shown in Fig. 1.

III. PAPR DISTRIBUTION OF PERIODIC INPUT SYMBOL SEQUENCES

Let $X[n]$ be an input symbol sequence of length N, which has nonzero complex values in the interval $0 \le n \le N/M - 1$ as follows:

$$X[n] = \begin{cases} \text{nonzero}, & 0 \le n \le N/M - 1 \\ 0, & \text{otherwise.} \end{cases}$$

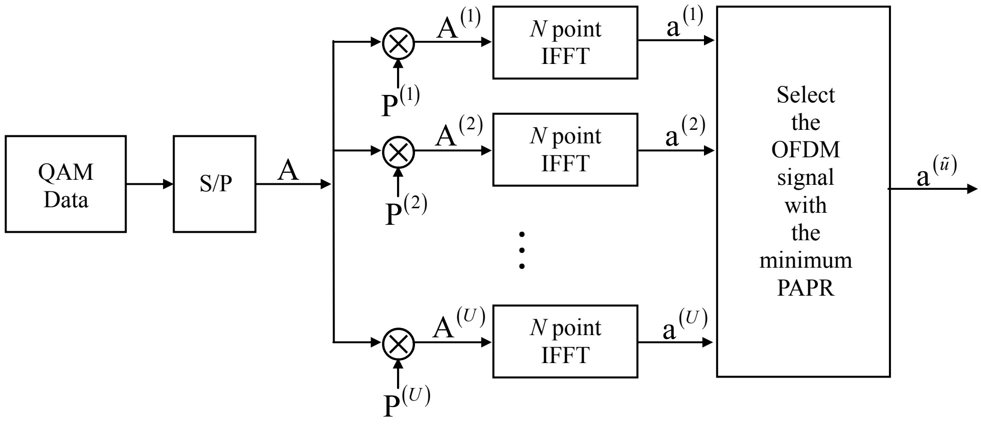

Fig. 1. Block diagram of SLM OFDM system.

If a sequence $A[n]$ of length N is generated by repeating the non-zero interval of $X[n]$ M times, $A[n]$ is expressed as

$$A[n]=\sum_{m=0}^{M-1} X\left[n-\frac{N}{M}m\right],\ 0\le n\le N-1.$$

Let $a[t]$ and $x[t]$ denote discrete time domain OFDM signals of $A[n]$ and $X[n]$, respectively. According to the shift property of Fourier transform, $x[t]$ is expressed as

$$a[t]=\sum_{m=0}^{M-1} x[t]\, e^{j2\pi\frac{m}{M}t}=\begin{cases} Mx[t], & t=0 \bmod M \\ 0, & \text{otherwise.}\end{cases} \tag{2}$$

In [1], the PAPR distribution of $a[t]$ with $M=1$ is evaluated. We will derive the PAPR distribution for $M\ge 1$. Before evaluating the PAPR distribution of $a[t]$, we first calculate the distribution of instantaneous power of $x[t]$. As $X[n]$ has nonzero value only within the interval $0\le n\le N/M-1$, $x[t]$ is given as

$$x[t]=\frac{1}{\sqrt{N}}\sum_{n=0}^{N/M-1} X[n]e^{j2\pi\frac{n}{N}t}$$

where nonzero $X[n]$ can be thought as independent zero-mean random variable with variance σ_X^2. The average power σ_x^2 of OFDM signal $x[t]$ is calculated using Parseval's theorem as follows:

$$\sigma_x^2 \triangleq E\left\{\left|x[t]\right|^2\right\}=\frac{N/M}{N}\sigma_X^2=\frac{\sigma_X^2}{M}. \tag{3}$$

Since $x[t]$ is the sum of N/M nonzero $X[n]$, $x[t]$ can be approximated as a zero-mean complex Gaussian distributed random variable with variance σ_x^2 due to the central limit theorem for large N/M. $r=\left|x[t]\right|$ has Rayleigh distribution and its probability density function (PDF) is written as

$$p_r(r)=\frac{2r}{\sigma_x^2}\exp\left(-\frac{r^2}{\sigma_x^2}\right). \tag{4}$$

Using (4) and $u=\left|a[t]\right|=Mr$ for $t=0 \bmod M$, u also has Rayleigh distribution and its PDF is expressed as

$$p_u(u)=\frac{2u}{M^2\sigma_x^2}\exp\left(-\frac{u^2}{M^2\sigma_x^2}\right). \tag{5}$$

Using (3) and (5), the probability that the magnitude of one OFDM signal sample $a[t]$ does not exceed a_0 is given by

$$\Pr\left\{|a|\le a_0\right\}=1-\exp\left(-\frac{a_0^2}{M\sigma_X^2}\right). \tag{6}$$

Suppose that $a[t]$ is statistically independent. Using (2) and (6), the probability that at least one magnitude value of OFDM signal $a[t]$ exceeds threshold a_0 can be approximated as

$$\begin{aligned}&\Pr\left\{\exists\, t,\ 0\le t\le N-1,\ \text{such that } \left|a[t]\right|>a_0\right\}\\ &=1-\Pr\left\{\max_{0\le t\le N-1}\left|a[t]\right|\le a_0\right\}\\ &=1-\left(1-\exp\left(-\frac{a_0^2}{M\sigma_X^2}\right)\right)^{N/M}.\end{aligned} \tag{7}$$

Using Parseval's theorem, the average power σ_a^2 of OFDM signal $a[t]$ is calculated as

$$\sigma_a^2 \triangleq E\left\{\left|a[t]\right|^2\right\}=\sigma_X^2. \tag{8}$$

Plugging (8) into (7), the probability $P_\zeta(\zeta_0)$ that the crest factor exceeds some threshold value $\zeta_0=a_0/\sigma_a$ is written as

$$P_\zeta(\zeta_0)=\Pr\{\zeta>\zeta_0\}=1-\left(1-\exp\left(-\frac{\zeta_0^2}{M}\right)\right)^{N/M}. \tag{9}$$

Fig. 2 shows the complementary cumulative distribution function (CCDF) of the PAPR obtained from (9). The probability that the OFDM signal exceeds the value $\mathrm{PAPR}_0=\zeta_0^2$ becomes higher as M increases.

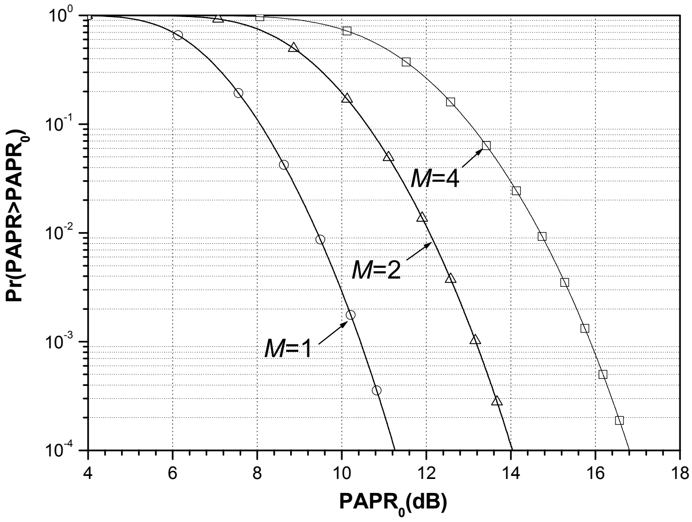

Fig. 2. CCDF of the PAPR of SLM OFDM system for N = 256 with $M \in \{1,2,4\}$.

IV. PAPR Bound of SLM OFDM Signals

Let $A[n]$ be an input symbol sequence of length N with arbitrary constellation such as M-PSK or M-QAM and $P[n]$ be a binary sequence with Hamming weight D. If we generate $B[n]$ by multiplying $A[n]$ with $(-1)^{P[n]}$, $B[n]-A[n]$ is written as

$$\begin{aligned} B[n]-A[n] &= A[n]\left\{(-1)^{P[n]}-1\right\} \\ &= -2\sum_{d=0}^{D-1} A[n]\delta[n-n_d] \end{aligned} \quad (10)$$

where $\delta[n]$ is the discrete-time delta function defined as

$$\delta[n] = \begin{cases} 1, & \text{if } n=0 \\ 0, & \text{otherwise} \end{cases}$$

and n_d, $0 \le n_d \le N-1$, denotes the index at which $P[n]$ is one. From (10), $\text{IFFT}\{B[n]-A[n]\}$ is given as

$$\text{IFFT}\{B[n]-A[n]\} = -\frac{2}{\sqrt{N}}\sum_{d=0}^{D-1} A[n_d]e^{j2\pi\frac{n_d}{N}t}. \quad (11)$$

Let $a[t]$ and $b[t]$ denote discrete time domain OFDM signals of the input symbol sequences $A[n]$ and $B[n]$, respectively. Then, $\left|b[t]-a[t]\right|$ is bounded as

$$\begin{aligned} \left|b[t]-a[t]\right| &\le \frac{2}{\sqrt{N}}\sum_{d=0}^{D-1}\left|A[n_d]e^{j2\pi\frac{n_d}{N}t}\right| \\ &= \frac{2}{\sqrt{N}}\sum_{d=0}^{D-1}\left|A[n_d]\right|. \end{aligned} \quad (12)$$

Using (12), we can derive the upper and lower bound of $\left|b[t]\right|$ as

$$\begin{aligned} \left|a[t]\right| - \frac{2}{\sqrt{N}}\sum_{d=0}^{D-1}\left|A[n_d]\right| &\le \left|b[t]\right| \\ &\le \left|a[t]\right| + \frac{2}{\sqrt{N}}\sum_{d=0}^{D-1}\left|A[n_d]\right|. \end{aligned} \quad (13)$$

Then, the crest factor ζ_b of $b[t]$ can be bounded as

$$\begin{aligned} \zeta_a - \frac{2}{\sigma\sqrt{N}}\sum_{d=0}^{D-1}\left|A[n_d]\right| &\le \zeta_b \\ &\le \zeta_a + \frac{2}{\sigma\sqrt{N}}\sum_{d=0}^{D-1}\left|A[n_d]\right| \end{aligned} \quad (14)$$

where ζ_a is the crest factor of $a[t]$ and σ^2 is the average power of $a[t]$ and $b[t]$. When $A[n]$ is M-PSK symbol of average power of unity, $a[t]$ has an average power of unity from Parseval's theorem. Then, (14) is simplified as

$$\zeta_a - \frac{2D}{\sqrt{N}} \le \zeta_b \le \zeta_a + \frac{2D}{\sqrt{N}}. \quad (15)$$

Since D is the Hamming distance of two sequences $A[n]$ and $B[n]$, $(N-D)$ is the Hamming distance of two sequences $-A[n]$ and $B[n]$. The OFDM signals of $A[n]$ and $-A[n]$ have the same crest factor. The crest factor ζ_b can be bounded as

$$\zeta_a - \frac{2(N-D)}{\sqrt{N}} \le \zeta_b \le \zeta_a + \frac{2(N-D)}{\sqrt{N}}. \quad (16)$$

From (15) and (16), the crest factors might be maximally apart from each other (independent) when $D = N/2$. This means that the phase sequences should be balanced and orthogonal to generate U statistically independent alternative OFDM input symbol sequences.

V. Phase Sequences for SLM OFDM System

It has been used without any mathematical proof that the phase sequences should be orthogonal for generating statistically independent alternative OFDM symbol sequences from an input symbol sequence.

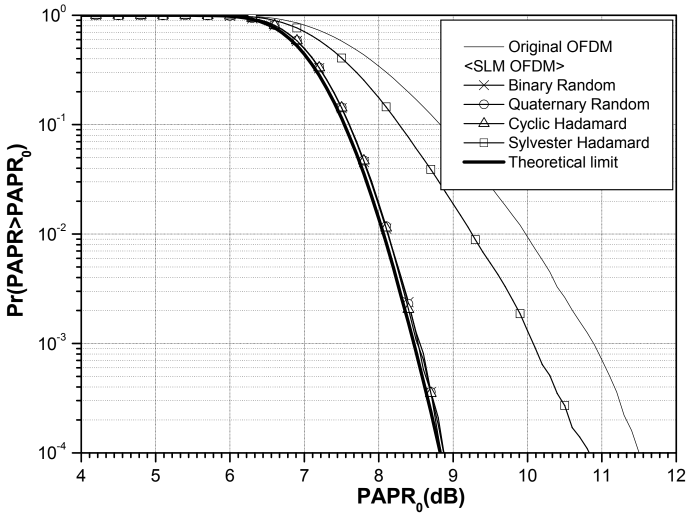

Fig. 3. Complementary cumulative distribution function of the PAPR of SLM OFDM system for various phase sequences when *N* = 256.

There has been no criterion to generate the optimal phase sequences for SLM OFDM system. From the two previous sections, we can conclude that the optimal phase sequences for SLM OFDM system should have the orthogonality and no periodicity.

Fig. 3 shows the CCDF of the PAPR of SLM ODFM system for some phase sequences, where the number of phase sequences is 4. In Fig. 3, a random sequence indicates the sequence that is generated randomly with uniform distribution. Random sequences achieve the theoretical limit for PAPR reduction of the SLM OFDM system. But the performance of Sylvester Hadamard sequences is not so good as that of random sequences though Sylvester Hadamard sequences are orthogonal.

The reason why Sylvester Hadamard sequences do not have the good performance in PAPR reduction is that they have periodicity. In Section III and IV, we have shown that if an input symbol sequence of OFDM system is periodic or similar to the periodic sequence, its OFDM signal has the large PAPR with high probability. If a random phase sequence is multiplied with the periodic input symbol sequence, the obtained sequence may get aperiodic property and its OFDM signal has the low PAPR with high probability. If we generate sequences randomly with uniform distribution, the random sequences may attain the near orthogonality and near aperiodic property to some degree. Hence, random sequences as a phase sequence can show good performance in PAPR reduction.

The rows of cyclic Hadamard matrix constructed by m-sequence are orthogonal and have no periodicity except the first row. Thus, the rows of cyclic Hadamard matrix might be the near optimal phase sequences for SLM OFDM system.

VI. Conclusion

We have derived two conditions for the optimal phase sequence set of the SLM OFDM system. First, the phase sequences are to be orthogonal each other. Second, the phase sequences should not be periodic or similar to periodic sequences. The rows of a cyclic Hadamard matrix constructed by an m-sequence meet the above two conditions.

Although the two conditions can not be said to be sufficient conditions for the optimal phase sequence set, it is shown from the simulation results that cyclic Hadamard sequence achieves the theoretical limit for PAPR reduction of the SLM OFDM system within 0.1dB. Thus, we propose cyclic Hadamard sequences as the near optimal phase sequence set for the SLM OFDM system.

References

[1] S. H. Müller, R. W. Bäuml, R F. H Fischer, and Johannes B. Huber, “OFDM with reduced peak-to-average power ratio by multiple signal representation,” in *Annals of Telecommun.*, vol. 52, no. 1-2, pp. 58-67, Feb. 1997.

[2] P. V. Eetvelt, G.. Wade and M. Tomlinson, “Peak to average power reducing for OFDM schemes by selective scrambling,” *Electron. Lett.*, vol. 32, no. 21, pp. 1963-1964, Oct. 1996.

[3] N. Ohkubo and T. Ohtsuki, “A peak to average power ratio reduction of multicarrier CDMA using selected mapping,” in *Proc. IEEE Veh. Technol. Conf.*, 2002.

[4] M. Breiling, S. H. Müller and J. B. Huber, “SLM peak power reduction without explicit side information,” *IEEE Commun. Lett.*, vol. 5, no. 6, pp. 239-241, June 2001.