Extension of LCZ Sequence Sets[1]

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Abstract

In this paper, given a q-ary LCZ sequence set with parameters (N, M, L, ϵ) and even q, we construct another q-ary LCZ sequence set with parameters $(2N, 2M, L, 2\epsilon)$ when q is even. Especially for odd L such that $L|N$, if a q-ary optimal LCZ sequence set with parameters $(N, M, L, 1)$ is used, the new set with parameters $(2N, 2M, L, 2)$ can be optimal in terms of the set size.

1. Introduction

In the quasi-synchronous code-division multiple-access (QS-CDMA) systems, time delay within a few chips among different users is allowed, which gives more flexibility in designing the wireless communication systems. In the design of a sequence set for QS-CDMA system, what matters most is to have low correlation zone (LCZ) around the origin rather than to minimize the overall maximum nontrivial correlation value [5]. In fact, LCZ sequences with smaller correlation magnitude within the zone show better performance than other well-known sequence families with optimal correlation property [5].

Let S be a set of M sequences of period N. If the magnitude of correlation function between any two sequences in S takes the values less than or equal to ϵ within the range $-L < \tau < L$, of the offset τ, then S is called an (N, M, L, ϵ) LCZ sequence set.

In this paper, given a q-ary LCZ sequence set with parameters (N, M, L, ϵ), we construct another q-ary LCZ sequence set with parameters $(2N, 2M, L, 2\epsilon)$ when q is even. Especially when $\epsilon = 1$, the new set can be optimal in terms of the set size.

2. Preliminaries

Let $R_{i,j}(\tau)$ be the correlation function between $s_i(t)$ and $s_j(t)$ of period N given as

$$R_{i,j}(\tau) = \sum_{t=0}^{N-1} \omega^{s_i(t) - s_j(t+\tau)}.$$

Let E_i be the characteristic set of the sequence $s_i(t)$, i.e., $\{t \in Z_N \mid s_i(t) = 1\}$. Let $d_{i,j}(\tau) = |E_i \cap (E_j + \tau)|$, where $\tau \in Z_N$, $E_j + \tau = \{c + \tau \mid c \in E_j\}$, and $+$ means addition modulo N. Then we can easily check the following lemma.

Lemma 1 The correlation function $R_{i,j}(\tau)$ can be expressed as

$$R_{i,j}(\tau) = N - 2(|E_i| + |E_j| - 2d_{i,j}(\tau)).$$

□

By the Chinese remainder theorem, Z_{2N}, the set of integers modulo $2N$ can be decomposed, for an odd integer N as $Z_{2N} \simeq Z_2 \otimes Z_N$ under the isomorphism $\phi\ :\ \tau \mapsto (\tau \bmod 2, \tau \bmod N)$.

We will check the optimality of our LCZ sequence sets with respect to the following bound.

Theorem 2 [Tang, Fan, and Matsufuji [8]] Let $\mathcal{S}$ be an LCZ sequence sets with parameters (N, M, L, ϵ). Then we have

$$ML - 1 \leq \frac{N-1}{1 - \epsilon^2/N}. \tag{1}$$

□

Especially for $\epsilon = 2$ and $n \geq 4$, (1) becomes

$$ML \leq N + 4.$$

3. Construction Method 1

In this section, we propose the construction method of a new near optimal LCZ sequence sets with parameters $(2N, 2M, L, 2\epsilon)$ or $(2N, 2M, L-1, 2\epsilon)$ from an optimal q-ary LCZ sequence set with parameters (N, M, L, ϵ) for an even integer q.

Let q be an even integer. Let $\mathcal{L}_1$ be a q-ary LCZ sequence set with parameters (N, M, L, ϵ) given as

$$\mathcal{L}_1 = \{v_i(t) \mid 0 \leq i \leq M-1,\ 0 \leq t \leq N-1\}.$$

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We will call $\mathcal{L}_1$ the component LCZ sequence set. Using the component LCZ sequence set, we can construct a new LCZ sequence set with twice the size and period.

Construction 1: Let $\mathcal{T}_1$ be the set of sequences given as

$$\mathcal{T}_1 = \{s_i(t) | 0 \le i \le 2M-1,\ 0 \le t \le 2N-1\}$$

where $s_i(t)$ is defined as

$$s_i(2t) = \begin{cases} v_i(t), & 0 \le i \le M-1 \\ v_{i-M}(t) + q/2, & M \le i \le 2M-1 \end{cases}$$

$$s_i(2t+1) = \begin{cases} v_i\left(t + \left\lfloor \frac{L}{2} \right\rfloor\right), & 0 \le i \le M-1 \\ v_{i-M}\left(t + \left\lfloor \frac{L}{2} \right\rfloor\right), & M \le i \le 2M-1 \end{cases}$$

where $\lfloor x \rfloor$ is the greatest integer not exceed x. □

Theorem 3 $\mathcal{T}_1$ in Construction 1 is an LCZ sequence set with parameters $(2N, 2M, L, 2\epsilon)$ if L is odd and with parameters $(2N, 2M, L-1, 2\epsilon)$ if L is even.

Proof : We will prove the case when L is odd. The case of even L can be proven similarly. Then we must consider the following six cases.

Case 1) $0 \le i, j \le M-1$ and τ is even;
In this case, $R_{i,j}(\tau)$ can be rewritten as

$$R_{i,j}(\tau) = \sum_{t=0}^{N-1} \omega^{v_i(t) - v_j(t + \frac{\tau}{2})} + \sum_{t=0}^{N-1} \omega^{v_i(t + \frac{L-1}{2}) - v_j(t + \frac{L-1}{2} + \frac{\tau}{2})}. \quad (2)$$

From the property of the LCZ sequence set with parameters (N, M, L, ϵ), it is clear that the magnitude of each summation in (2) is less than or equal to ϵ within the range $-2L < \tau < 2L$. Therefore, $|R_{i,j}(\tau)| \le 2\epsilon$ within the range $-2L < \tau < 2L$.

Case 2) $0 \le i, j \le M-1$ and τ is odd;
In this case, $R_{i,j}(\tau)$ can be rewritten as

$$R_{i,j}(\tau) = \sum_{t=0}^{N-1} \omega^{v_i(t) - v_j(t + \frac{L-1}{2} + \frac{\tau+1}{2})} + \sum_{t=0}^{N-1} \omega^{v_i(t + \frac{L-1}{2}) - v_j(t + \frac{\tau-1}{2})}. \quad (3)$$

From the property of the LCZ sequence set with parameters (N, M, L, ϵ) and in-phase autocorrelation, it is clear that the magnitude of the first summation in (3) is less than or equal to ϵ within the range $-3L < \tau < L$. And it is also clear that the magnitude of the second summation in (3) is less than or equal to ϵ within the range $-L < \tau < 3L$. Therefore, $|R_{i,j}(\tau)| \le 2\epsilon$ within the range $-L < \tau < L$.

Case 3) $0 \le i \le M-1$, $M \le j \le 2M-1$ (or $M \le i \le 2M-1$, $0 \le j \le M-1$), and τ is even;
It is easy to see that $\omega^{-q/2} = -1$. Therefore, $R_{i,j}(\tau)$ can be rewritten as

$$\begin{aligned} R_{i,j}(\tau) &= \sum_{t=0}^{N-1} \omega^{v_i(t) - v_{j-M}(t + \frac{\tau}{2})} \\ &\quad + \sum_{t=0}^{N-1} \omega^{v_i(t + \frac{L-1}{2}) - v_{j-M}(t + \frac{L-1}{2} + \frac{\tau}{2}) - \frac{q}{2}} \\ &= \sum_{t=0}^{N-1} \omega^{v_i(t) - v_{j-M}(t + \frac{\tau}{2})} \\ &\quad - \sum_{t=0}^{N-1} \omega^{v_i(t + \frac{L-1}{2}) - v_{j-M}(t + \frac{L-1}{2} + \frac{\tau}{2})}. \end{aligned}$$

Similarly to Case 1), $|R_{i,j}(\tau)| \le 2\epsilon$ within the range $-2L < \tau < 2L$.

Case 4) $0 \le i \le M-1$, $M \le j \le 2M-1$ (or $M \le i \le 2M-1$, $0 \le j \le M-1$), and τ is odd;
It is easy to see that $\omega^{-q/2} = -1$. Therefore $R_{i,j}(\tau)$ can be rewritten as

$$\begin{aligned} R_{i,j}(\tau) &= \sum_{t=0}^{N-1} \omega^{v_i(t) - v_{j-M}(t + \frac{L-1}{2} + \frac{\tau+1}{2}) - \frac{q}{2}} \\ &\quad + \sum_{t=0}^{N-1} \omega^{v_i(t + \frac{L-1}{2}) - v_{j-M}(t + \frac{\tau-1}{2})} \\ &= -\sum_{t=0}^{N-1} \omega^{v_i(t) - v_{j-M}(t + \frac{L-1}{2} + \frac{\tau+1}{2})} \\ &\quad + \sum_{t=0}^{N-1} \omega^{v_i(t + \frac{L-1}{2}) - v_{j-M}(t + \frac{\tau-1}{2})}. \end{aligned}$$

Similarly to Case 2), $|R_{i,j}(\tau)| \le 2\epsilon$ within the range $-L < \tau < L$.

Case 5) $M \le i, j \le 2M-1$ and τ is even;
In this case, $R_{i,j}(\tau)$ can be rewritten as

$$R_{i,j}(\tau) = \sum_{t=0}^{N-1} \omega^{v_{i-M}(t) - v_{j-M}(t + \frac{\tau}{2})} + \sum_{t=0}^{N-1} \omega^{v_{i-M}(t + \frac{L-1}{2}) - v_{j-M}(t + \frac{L-1}{2} + \frac{\tau}{2})}.$$

Similarly to Case 1), $|R_{i,j}(\tau)| \le 2\epsilon$ within the range $-2L < \tau < 2L$.

Case 6) $M \le i, j \le 2M-1$ and τ is odd;
In this case, $R_{i,j}(\tau)$ can be rewritten as

$$R_{i,j}(\tau) = \sum_{t=0}^{N-1} \omega^{v_{i-M}(t) - v_{j-M}(t + \frac{L-1}{2} + \frac{\tau+1}{2})} + \sum_{t=0}^{N-1} \omega^{v_{i-M}(t + \frac{L-1}{2}) - v_{j-M}(t + \frac{\tau-1}{2})}.$$

Similarly to Case 2), $|R_{i,j}(\tau)| \le 2\epsilon$ within the range $-L < \tau < L$.

From the above 6 cases, it is clear that $\mathcal{T}_1$ is an LCZ sequence set with parameters $(2N, 2M, L, 2\epsilon)$, if L is odd.

□

To the best of our knowledge, the only known optimal q-ary LCZ sequence sets with parameters $(N, M, L, 1)$ and even q are due to [2] for $q = 2$ and [3] for $q = 4$.

When we apply Construction 1 to the above optimal binary and quaternary LCZ sequence sets, it turns out that resulting LCZ sequence sets with parameters $(2N, 2M, L, 2)$ become optimal.

Example 4 Let $n = 4$, $e = m = 2$, and $T = (2^n - 1)/(2^e - 1) = 5$. Let α be a primitive element in finite field F_{2^4} with 16 elements. Let L_1 be the quaternary LCZ set $\mathcal{L}_1$ with parameter $(15, 3, 5, 1)$ given as

$$\mathcal{L}_1 = \{m_i(t) | 0 \le i \le 2,\ 0 \le t < 15\}$$

$$m_i(t) = \begin{cases} 2\mathrm{tr}_1^4(\alpha^t), & \text{for } i = 0 \\ \mathrm{tr}_1^4(\alpha^t) \boxplus 2\mathrm{tr}_1^4(\alpha^{t+5i}), & \text{otherwise} \end{cases}$$

where $\boxplus$ denotes addition modulo 4 and $\mathrm{tr}_1^4(\cdot)$ is a trace function from F_{2^4} to F_2.

Then the following $\mathcal{T}_1$ is the optimal quaternary LCZ sequence set with parameters $(30, 6, 5, 2)$

$$\mathcal{T}_1 = \{s_i(t) | 0 \le i \le 5,\ 0 \le t < 30\}$$

where $s_i(t)$ is given as

$$\begin{aligned}
s_0(t) &= 000200200202220222002202222222020 \\
s_1(t) &= 022122102303323320310113113012 \\
s_2(t) &= 022322302101121120130331331032 \\
s_3(t) &= 202220002222000220022202020000 \\
s_4(t) &= 002320122101303122330311133210 \\
s_5(t) &= 002120322303101322110133311230.
\end{aligned}$$

□

4. Construction Method 2

In this section, we would like to propose the other construction scheme of LCZ sequence sets. Let $\mathcal{L}_2$ be a set of the binary LCZ sequence set with parameters $(N, M, L, 1)$ where $N \equiv 3 \bmod 4$ and $L|N$. Assume that the correlation functions $R_{i,j}(\tau)$ between any two sequences $v_i(t)$ and $v_j(t)$ in $\mathcal{L}_2$ take the value -1 all the time except for $\tau \equiv 0 \bmod L(\tau \ne 0)$. Also assume that all the sequences in $\mathcal{L}_2$ are balanced.

Let D_i be the characteristic set of the LCZ sequence $v_i(t)$ in $\mathcal{L}_2$. Define two subsets $E_i^{(1)}$ and $E_i^{(2)}$ of $Z_2 \otimes Z_N$ as

$$\begin{aligned}
E_i^{(1)} &= \{0\} \otimes D_i \cup \{1\} \otimes (D_i + L) \\
E_i^{(2)} &= \{0\} \otimes D_i \cup \{1\} \otimes (\overline{D}_i + L).
\end{aligned}$$

Let $\mathcal{V}_k$, $k = 1, 2$, be the set of all the characteristic sequences $s_i^{(k)}(t)$ of the sets $E_i^{(k)}$, $i = 0, 1, \cdots, M-1$, respectively.

Construction 2: The new binary LCZ sequence set is defined as

$$\begin{aligned}
\mathcal{T}_2 &= \mathcal{V}_1 \cup \mathcal{V}_2 \\
&= \{s_i^{(k)}(t) \mid 0 \le i \le M-1,\ 0 \le t \le 2N-1, \\
&\qquad k = 0, 1\}
\end{aligned}$$

where $s_i^{(k)}(t) \in \mathcal{V}_k$.

□

We can obtain the set size and LCZ of $\mathcal{T}_2$ as the following theorem.

Theorem 5 $\mathcal{T}_2 = \mathcal{V}_1 \cup \mathcal{V}_2$ is the binary LCZ sequence set with parameters $(2N, 2M, L, 2)$.

Proof: Let $\tau = (\tau_1, \tau_2) \in Z_2 \otimes Z_N$. We have to consider three cases. First, for the case of $s_i^{(1)}(t)$ and $s_j^{(1)}(t)$, we have

$$\begin{aligned}
d_{i,j}(\tau) &= |E_i^{(1)} \cap (E_j^{(1)} + \tau)| \\
&= |(\{0\} \otimes D_i \cup \{1\} \otimes (D_i + L)) \cap \\
&\qquad (\{\tau_1\} \otimes (D_j + \tau_2) \cup \{1 + \tau_1\} \otimes (D_j + L + \tau_2))| \\
&= |\{0\} \cap \{\tau_1\}||D_i \cap (D_j + \tau_2)| \\
&\quad + |\{0\} \cap \{1 + \tau_1\}||D_i \cap (D_j + L + \tau_2)| \\
&\quad + |\{1\} \cap \{\tau_1\}||(D_i + L) \cap (D_j + \tau_2)| \\
&\quad + |\{1\} \cap \{1 + \tau_1\}||(D_i + L) \cap (D_j + L + \tau_2)| \\
&= \begin{cases} |D_i \cap (D_j + \tau_2)| + |(D_i + L) \cap (D_j + L + \tau_2)|, \\ \hfill \text{if } \tau_1 = 0, \\ |D_i \cap (D_j + L + \tau_2)| + |(D_i + L) \cap (D_j + \tau_2)|, \\ \hfill \text{if } \tau_1 = 1. \end{cases}
\end{aligned}$$

From the balance property and correlation value -1 of the component LCZ sequence set for $\tau \not\equiv 0 \bmod L$, we have

$$|D_i \cap (D_j + \tau_2)| = |D_i \cap (D_j + L + \tau_2)| = \frac{(N+1)}{4}$$

$$\begin{aligned}
|(D_i + L) \cap (D_j + L + \tau_2)| &= |(D_i + L) \cap (D_j + \tau_2)| \\
&= \frac{(N+1)}{4}
\end{aligned}$$

for $\tau_2 \not\equiv 0 \bmod L$. Thus we have

$$\begin{aligned}
d_{i,j}(\tau) &= \frac{(N+1)}{2} \\
R_{i,j}(\tau) &= -2
\end{aligned}$$

at $\tau \not\equiv 0 \bmod L$. When $\tau = 0$, we have

$$d_{i,j}(\tau) = |D_i \cap (D_j)| + |(D_i + L) \cap (D_j + L)|.$$

Similarly, it is clear that

$$\begin{aligned}
|D_i \cap D_j| &= |(D_i + L) \cap (\overline{D}_j + L)| \\
&= \frac{(N+1)}{4}
\end{aligned}$$

for $i \neq j$. Therefore we also have

$$d_{i,j}(0) = \frac{(N+1)}{2}$$
$$R_{i,j}(0) = -2.$$

Second, for the case of $s_i^{(1)}(t)$ and $s_j^{(2)}(t)$, we have

$$d_{i,j}(\tau) = |E_i^{(1)} \cap (E_j^{(2)} + \tau)|$$
$$= \begin{cases} |D_i \cap (D_j + \tau_2)| + |(D_i + L) \cap (\overline{D}_j + L + \tau_2)|, \\ \hfill \text{if } \tau_1 = 0, \\ |D_i \cap (\overline{D}_j + L + \tau_2)| + |(D_i + L) \cap (D_j + \tau_2)|, \\ \hfill \text{if } \tau_1 = 1. \end{cases}$$

Similarly to the first case, we have

$$|D_i \cap (D_j + \tau_2)| = |D_i \cap (\overline{D}_j + L + \tau_2)| = \frac{(N+1)}{4}$$

$$|(D_i + L) \cap (\overline{D}_j + L + \tau_2)| = |(D_i + L) \cap (D_j + \tau_2)| = \frac{(N+1)}{4}$$

for $\tau_2 \not\equiv 0 \bmod L$, which yields

$$d_{i,j}(\tau) = \frac{(N+1)}{2}$$
$$R_{i,j}(\tau) = 0.$$

For $\tau = 0$, it is easy to derive $R_{i,j}(0) = 0$.

Finally, for the case of $s_i^{(2)}(t)$ and $s_j^{(2)}(t)$, we have

$$d_{i,j}(\tau) = |E_i^{(2)} \cap (E_j^{(2)} + \tau)|$$
$$= \begin{cases} |D_i \cap (D_j + \tau_2)| + |(\overline{D}_i + L) \cap (\overline{D}_j + L + \tau_2)|, \\ \hfill \text{if } \tau_1 = 0, \\ |D_i \cap (\overline{D}_j + L + \tau_2)| + |(\overline{D}_i + L) \cap (D_j + \tau_2)|, \\ \hfill \text{if } \tau_1 = 1. \end{cases}$$

Similarly to the previous cases, it is easy to compute $R_{i,j}(\tau) = \pm 2$ except for the in-phase autocorrelation for $-L < \tau < L$.

□

We can easily obtain the following corollary.

Corollary 6 $\mathcal{T}_2$ is optimal if and only if the component binary LCZ sequence set $\mathcal{L}_2$ is optimal.

Proof: From Theorem 2 and the fact that L divides N, it is clear that the component LCZ sequence sets and their extended LCZ sequence sets are optimal if and only if $N = LM$. Therefore $\mathcal{T}_2$ is optimal if and only if the component binary LCZ sequence set $\mathcal{L}_2$ is optimal.

□

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