Exact Bit Error Probability of Orthogonal Space-Time Block Codes with QAM[1]

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Abstract — **In this paper, for homogeneous and nonhomogeneous linear orthogonal space-time codes with square QAM's, the exact expressions of the bit error probability in slow Rayleigh fading channel are derived.**

I. Introduction

Let L_t and L_r be the numbers of transmit antennas and receive antennas, respectively. The codeword $\mathbf{X}$ is a matrix with N rows and L_t columns. At time n, the L_t symbols in each row of $\mathbf{X}$ are simultaneously transmitted via the L_t transmit antennas, where $1 \le n \le N$. We assume slow Rayleigh fading channel, where fading is assumed to be constant over the duration of a codeword matrix. Let $\mathbf{A}$ be an $L_t \times L_r$ channel matrix and known to the receiver. Let $\mathbf{Y}$ be a $N \times L_r$ received matrix and $\mathbf{W}$ a $N \times L_r$ noise matrix. Then we have

$$\mathbf{Y} = \sqrt{\frac{\overline{\rho}}{E_m}}\mathbf{X} \cdot \mathbf{A} + \mathbf{W}$$

where $\overline{\rho}$ is the average signal to noise ratio (SNR) and E_m the average energy transmitted from all L_t transmit antennas combined during a symbol period.

The message vector of length L_s is denoted by

$$\mathbf{s} = (s_1, s_2, \cdots, s_{L_s}). \tag{1}$$

Since QAM is defined in the two-dimensional signal space, it is necessary to split the L_s-dimensional complex message vector in (1) into the $2L_s$-dimensional real vector given by

$$\mathbf{s}' = (s_{1,x}, s_{1,y}, s_{2,x}, s_{2,y}, \cdots, s_{L_s,x}, s_{L_s,y}) \tag{2}$$

where $s_k = s_{k,x} + js_{k,y}, 1 \le k \le L_s$.

Let $\mathcal{C}(\cdot)$ be a mapping from an L_s-tuple complex message vector to the columnwise orthogonal $N \times L_t$ codeword matrix given by $\mathbf{X} = \mathcal{C}(\mathbf{s})$ [1]. By using the columnwise orthogonality of the orthogonal space-time codes, we have

$$\mathcal{C}(\mathbf{s})^H\mathcal{C}(\mathbf{s}) = \mathrm{diag}\{\sum_{k=1}^{L_s} g_{k,1} \cdot |s_k|^2, \cdots, \sum_{k=1}^{L_s} g_{k,L_t} \cdot |s_k|^2\} \tag{3}$$

where $(\cdot)^H$ denotes the Hermitian operator. Note that the diagonal terms are the squared magnitudes of the columns of $\mathcal{C}(\mathbf{s})$ and $g_{k,i}$ is the multiplicity of the symbol $|s_k|^2$ in the i-th antenna.

Linear orthogonal space-time codes can be classified according to the values of $g_{k,i}$. A linear orthogonal space-time code is said to be *homogeneous* if $g_{k,i}$ is constant.

Let E_s be the average symbol energy of s_k. Thus E_m can be expressed as $E_m = \frac{1}{N}\sum_{i=1}^{L_t}\sum_{k=1}^{L_s} g_{k,i} \cdot E_s$.

Let $\mathcal{C}$ be a linear orthogonal space-time code. Let $\mathbf{p} = \{p_1, p_2, \cdots, p_{L_s}\}$ and $\mathbf{q} = \{q_1, q_2, \cdots, q_{L_s}\}$ be distinct message vectors and $\mathcal{C}(\mathbf{p})$ and $\mathcal{C}(\mathbf{q})$ be the corresponding codewords, respectively. Let $\mathbf{X} = \mathcal{C}(\mathbf{p})$ and $\hat{\mathbf{X}} = \mathcal{C}(\mathbf{q})$. From the columnwise orthogonality of $\mathcal{C}(\cdot)$, the difference matrix $\mathbf{X} - \hat{\mathbf{X}}$ also has a columnwise orthogonal property. Using (3), we can get the pairwise error probability

$$P(\mathcal{C}(\mathbf{p}) \to \mathcal{C}(\mathbf{q})) = \frac{1}{\pi}\int_0^{\pi/2}\prod_{i=1}^{L_t}\left[1 + \frac{\overline{\gamma}}{4\sin^2\theta}\sum_{k=1}^{L_s} g_{k,i}|p_k - q_k|^2\right]^{-L_r} d\theta.$$

where $\overline{\gamma} = \frac{\overline{\rho}}{E_m}$.

II. Exact Bit Error Probability for QAM's

Let $d_h(\mathbf{a}, \mathbf{b})$ be the Hamming distance between two vectors $\mathbf{a}$ and $\mathbf{b}$. Let $\mathbf{s}$ and $\hat{\mathbf{s}}$ be vectors such that $s_{k,x} \ne \hat{s}_{k,x}$ and $d_h(\mathbf{s}', \hat{\mathbf{s}}') = 1$ where $\mathbf{s}'$ and $\hat{\mathbf{s}}'$ are defined in (2). We can define the one-dimensional component error function as a function of distance $l = |s_{k,x} - \hat{s}_{k,x}|$:

$$P(\mathcal{C}(\mathbf{s}) \to \mathcal{C}(\hat{\mathbf{s}})) = \mathcal{Q}_{k,x}(l) = \frac{1}{\pi}\int_0^{\pi/2}\prod_{i=1}^{L_t}\left[1 + \frac{\overline{\gamma}}{4\sin^2\theta} g_{k,i} \cdot l^2\right]^{-L_r} d\theta.$$

It is clear that $\mathcal{Q}_{k,x}(l) = \mathcal{Q}_{k,y}(l)$. For homogeneous codes, it can be rewritten in closed form by the result of [2].

Let $2d$ be the smallest distance between two constellation points of given square QAM. By using the one-dimensional component error function, the exact bit error probability for homogeneouse and nonhomogeneouse orthogonal space-time codes are derived as

$$\begin{aligned}
P_{QPSK} &= \frac{1}{2L_s}\sum_{k=1}^{L_s}\{P_e(s_{k,x}) + P_e(s_{k,y})\} = \frac{1}{L_s}\sum_{k=1}^{L_s}\mathcal{Q}_{k,x}(2d) \\
P_{16QAM} &= \frac{1}{L_s}\sum_{k=1}^{L_S}\{\frac{3}{4}\mathcal{Q}_{k,x}(2d) + \frac{1}{2}\mathcal{Q}_{k,x}(6d) - \frac{1}{4}\mathcal{Q}_{k,x}(10d)\} \\
P_{64QAM} &= \frac{1}{L_s}\sum_{k=1}^{L_s}\frac{1}{12}\{7\mathcal{Q}_{k,x}(2d) + 6\mathcal{Q}_{k,x}(6d) - \mathcal{Q}_{k,x}(10d) \\
&\quad + \mathcal{Q}_{k,x}(18d) - \mathcal{Q}_{k,x}(26d)\} \\
P_{256QAM} &= \frac{1}{L_s}\sum_{k=1}^{L_s}\frac{1}{32}\{15\mathcal{Q}_{k,x}(2d) + 14\mathcal{Q}_{k,x}(6d) - \mathcal{Q}_{k,x}(10d) \\
&+ \quad 5\mathcal{Q}_{k,x}(18d) + 4\mathcal{Q}_{k,x}(22d) - 5\mathcal{Q}_{k,x}(26d) - 4\mathcal{Q}_{k,x}(30d) \\
&+ \quad 5\mathcal{Q}_{k,x}(34d) + 4\mathcal{Q}_{k,x}(38d) - 3\mathcal{Q}_{k,x}(42d) - 2\mathcal{Q}_{k,x}(46d) \\
&+ \quad \mathcal{Q}_{k,x}(50d) - \mathcal{Q}_{k,x}(58d).\}
\end{aligned}$$

References

[1] V. Tarokh, H. Jafarkhani, and A. R. Calderbank, "Space-time block codes from orthogonal designs," *IEEE Trans. Inform. Theory*, vol. 45, pp. 1456–1467, Jul. 1999.

[2] M. K. Simon, "Evaluation of average bit error probability for space-time coding based on a simpler exact evaluation of pairwise error probability," *J. Commun. Networks*, vol. 3, no. 3, Sept. 2001.

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