The 18th Annual IEEE International Symposium on Personal, Indoor and Mobile Radio Communications (PIMRC'07)

# CONSTRUCTION OF THE NEAR OPTIMAL PRT SET USING THE CYCLIC DIFFERENCE SET IN TR SCHEME

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## ABSTRACT

In the tone reservation (TR) scheme, it is known that the set of randomly selected peak reduction tones (PRT's) performs better than the contiguous PRT set and the interleaved PRT set in the PAPR reduction of orthogonal frequency division multiplexing (OFDM) signals. It is also known that finding the optimal PRT set is equivalent to solving the secondary peak minimization problem in the TR scheme. However, this problem cannot be solved for the practical number of tones because choosing $k$ tones out of $N$ subcarriers requires $O(k)$ computational complexity. In this paper, the near optimal PRT set of the TR scheme is proposed, which is constructed from the cyclic difference set.

## I. INTRODUCTION

Orthogonal frequency division multiplexing (OFDM) system has become the promising solution for the next generation wireless communication systems which require various high data rate services. Multiplexing a serial data symbol stream into a large number of orthogonal tones makes the OFDM signals bandwidth efficient [1]. It has been shown that the performance of OFDM system over frequency selective fading channels is better than that of the single carrier modulation system [2].

However, the major drawback of the OFDM system is its large envelope fluctuation. Since most practical systems are peak-power limited, operating the amplifier in the linear region often results in power inefficiency.

Recently, lots of works have been done in developing schemes to reduce the peak to average power ratio (PAPR) of OFDM signals. A simple and widely used scheme is clipping the signal to limit PAPR below a threshold level, but it causes both in-band distortion and out-of-band radiation [3]. Block coding [4], encoding input data into codewords with low PAPR, is another well-known PAPR reduction scheme, but it incurs the loss of transmission rate.

In [5], Tellado proposed tone reservation (TR) which reserves a small number of peak reduction tones (PRT's) to reduce the PAPR. The PAPR reduction performance of the TR increases in proportion to the number of reserved tones. It is known that the randomly generated PRT set performs better than the contiguous tone set and the interleaved tone set [6]. Finding the optimal PRT set is equivalent to finding the PRT set with the minimum secondary peak value of the time domain kernel, which is obtained by inverse fast Fourier-transforming (IFFT-ing) the characteristic sequence of the PRT set. The secondary peak minimization problem is known as NP-hard, which cannot be solved for the practical number of tones. In this paper, we construct the near optimal PRT set by modifying the cyclic difference set.

The rest of this paper is organized as follows. In Section II, the TR scheme is described. The construction method of the near optimal PRT set from the cyclic difference set are proposed in Section III and concluding remarks are given in Section IV.

## II. TONE RESERVATION

In the TR scheme, some tones are reserved to generate PAPR reduction signals and they are not used for data transmission [7]. Let $\mathcal{R} = \{i_0, i_1, \cdots, i_{W-1}\}$ denote the ordered set of the positions of the reserved tones and $\mathcal{R}^{\mathcal{C}}$ the complement set of $\mathcal{R}$ in $\mathcal{N} = \{0, 1, \cdots, N-1\}$, where $N$ is the number of tones of the OFDM signal and $W$ the number of reserved tones for PAPR reduction. $\mathcal{R}$ is also called a PRT set. The input symbol $A_k$ of the TR scheme is expressed as

$$A_k = X_k + C_k = \begin{cases} C_k, & k \in \mathcal{R} \\ X_k, & k \in \mathcal{R}^{\mathcal{C}} \end{cases}$$

where $X_k$ is a data symbol and $C_k$ a PAPR reduction symbol. Using the IFFT matrix $\mathbf{Q}$, the OFDM signal sequence in the time domain $\mathbf{a} = [a_0 a_1 \cdots a_{N-1}]^T$ is obtained as

$$\mathbf{a} = \mathbf{Q}(\mathbf{X} + \mathbf{C}) \quad (1)$$

where $\mathbf{X} = [X_0 X_1 \cdots X_{N-1}]^T$ and $\mathbf{C} = [C_0 C_1 \cdots C_{N-1}]^T$.

Let the data signal sequence $\mathbf{x}$ and the PAPR reduction signal sequence $\mathbf{c}$ be defined as $\mathbf{x} = [x_0 x_1 \cdots x_{N-1}]^T = \mathbf{QX}$ and $\mathbf{c} = [c_0 c_1 \cdots c_{N-1}]^T = \mathbf{QC}$, respectively. Since the Fourier transform is a linear operation, the OFDM signal sequence is the sum of the data signal sequence and the PAPR reduction signal sequence, that is, $\mathbf{a} = \mathbf{x} + \mathbf{c}$.

In [5], the PAPR of the OFDM signal sequence $\mathbf{a}$ is defined as

$$\text{PAPR}(\mathbf{a}) = \frac{\max\limits_{0 \le t \le N-1} |x_t + c_t|^2}{\text{E}[|x_t|^2]} \quad (2)$$

where $\text{E}[\cdot]$ is the expectation operator. The PAPR reduction signal sequence $\mathbf{c}$ can be iteratively obtained as follows. Let $\mathbf{p} = [p_0 p_1 \cdots p_{N-1}]^T$ be the time domain kernel defined as

$$\mathbf{p} = \mathbf{QP}$$

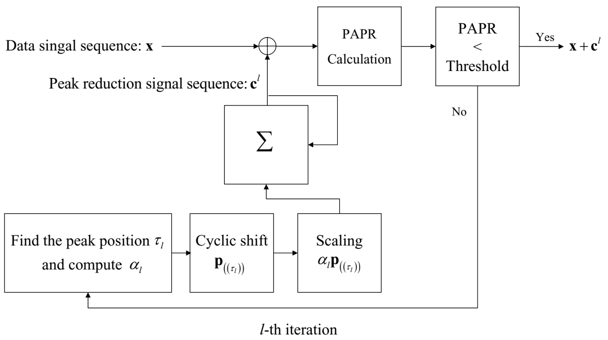


Figure 1: Relationship between the secondary peak and the variance of PRT sets when $N = 32$ and $W = 8$. All possible PRT sets are used.

where $\mathbf{P} = [P_0 P_1 \cdots P_{N-1}]^T$ is called a frequency domain kernel with binary $\{0, 1\}$ elements. Thus, the maximum peak of $\mathbf{p}$ is always $p_0$. The time domain kernel $\mathbf{p}$ is used to synthesize the PAPR reduction signal sequence $\mathbf{c}$, iteratively [6]. That is, the PAPR reduction signal sequence $\mathbf{c}^l$ at the $l$-th iteration is obtained as

$$\mathbf{c}^l = \sum_{i=1}^{l} \alpha_i \mathbf{p}_{((\tau_i))} \tag{3}$$

where $\mathbf{p}_{((\tau_i))}$ denotes a circular shift of $\mathbf{p}$ to the right by $\tau_i$ and $\alpha_i$ is a complex scaling factor. For simplicity, we assume that only one maximum peak of OFDM signal is reduced at each iteration in (3). The circular shift $\tau_i$ is determined as

$$\tau_i = \operatorname*{argmax}_{0 \le t \le N-1} |x_t + c_t^{i-1}|.$$

The complex scaling factor $\alpha_i$ is computed according to the threshold level $\zeta$ and the maximum peak value $u_i = x_{\tau_i} + c_{\tau_i}^{i-1}$ as

$$|u_i + \alpha_i p_0| = \zeta$$

where the threshold level affects the PAPR reduction performance, which is usually determined from the simulation results. If the maximum number of iterations is reached or the threshold level (desired level) is obtained, the iteration stops. From the shift property of the Fourier transform, it is guaranteed that $\mathbf{Q}^{-1}\mathbf{p}_{((\tau_i))}$ always has zero value in $\mathcal{R}^{\mathcal{C}}$. Fig. 1 shows the block diagram of the TR scheme.

The PAPR reduction performance depends on the time domain kernel $\mathbf{p}$ and the best performance can be achieved when the time domain kernel $\mathbf{p}$ is a discrete impulse because the maximum peak can be cancelled without affecting other signal samples at each iteration. But, in order for the time domain kernel $\mathbf{p}$ to be a discrete impulse, all the tones should be allocated to the PRT set. As the number of reserved tones becomes larger, the PAPR reduction performance is improved but the data transmission rate decreases. In general, the number of reserved tones is usually less than 15 percentage of $N$.

When the PRT set $\mathcal{R}$ is given, it is known that the optimal choice of $\mathbf{P}$ corresponds to minimizing the secondary peaks of $\mathbf{p}$. In this situation, it is shown in [6] that the optimal frequency domain kernel $\mathbf{P}$ is obtained as

$$P_k = \begin{cases} 1, & k \in \mathcal{R} \\ 0, & k \in \mathcal{R}^{\mathcal{C}}. \end{cases} \tag{4}$$

Then, the optimal frequency domain kernel $\mathbf{P}$ corresponds to the characteristic sequence of the PRT set $\mathcal{R}$ and the maximum peak of $\mathbf{p}$ is always $p_0 = |R|$ because $\mathbf{P}$ is a $\{0, 1\}$ sequence.

Clearly, the PAPR reduction performance of the TR scheme depends on the selection of the PRT set $\mathcal{R}$. Let $\mathcal{R}^{opt}$ be the optimal PRT set given by [6]

$$\mathcal{R}^{opt} = \operatorname*{argmin}_{\{i_o, i_1, \cdots, i_{W-1}\}} \|[|p_1||p_2|\cdots|p_{N-1}|]\|_\infty$$

where $\|\cdot\|_v$ denotes the $v$-norm and $\infty$-norm refers to the maximum value. From now on, we will assume that $\mathbf{p}$ is the time domain kernel obtained by IFFT-ing the characteristic sequence $\mathbf{P}$ of the PRT set $\mathcal{R}$.

It is known that this problem is NP-hard because the time domain kernel $\mathbf{p}$ must be optimized over all possible discrete sets $\mathcal{R}$ [6]. Thus, it cannot be solved for the practical values of $N$ and $W$. Since the computational complexity for evaluating the secondary peak of the time domain kernel $\mathbf{p}$ by IFFT-ing the characteristic sequence of the PRT set is very high, searching over all possible PRT sets takes too much time for large $N$ and practical value $W$. For example with $N = 128$ and $W = 10$, the number of candidate PRT sets is $\binom{128}{10} \simeq 2.27 \times 10^{14}$. This motivates us to find an efficient method to find a near optimal PRT set.

## III. Construction of the Near Optimal PRT Set Using Cyclic Difference Set

Let $\mathbf{y} = [y_0, y_1, y_2, \cdots, y_{N-1}]^T$, $\sum_{t=0}^{N-1} y_t = \gamma$, and $0 \le y_t \le y_0$, $1 \le t \le N-1$. Suppose that $y_0$ is a fixed value and $y_t$'s, $1 \le t \le N-1$, are variables. Then, we get

$$\max_{1 \le t \le N-1} y_t \ge \frac{1}{N-1} \sum_{t=1}^{N-1} y_t = \frac{\gamma - y_0}{N-1}.$$

In this case, it is clear that $\max_{1 \le t \le N-1} y_t$ is minimized when $y_1 = y_2 = \cdots = y_{N-1}$. The variance $\sigma_{\mathbf{y}}^2$ of $\mathbf{y}$ can be given as

$$\begin{aligned} \sigma_{\mathbf{y}}^2 &= \frac{1}{N} \sum_{t=0}^{N-1} y_t^2 - (\frac{1}{N} \sum_{t=0}^{N-1} y_t)^2 \\ &\ge \frac{1}{N(N-1)} (\sum_{t=1}^{N-1} y_t)^2 + \frac{1}{N} y_0^2 - (\frac{\gamma}{N})^2 \\ &= \frac{1}{N(N-1)} (\gamma - y_0)^2 + \frac{1}{N} {y_0}^2 - (\frac{\gamma}{N})^2 \end{aligned}$$

where the inequality holds from the Caushy-Schwartz inequality.

Thus, the variance $\sigma_{\mathbf{y}}^2$ can also be minimized when $y_1 = y_2 = \cdots = y_{N-1}$. Let $y_0 = |p_0|^2$ and $y_t = |p_t|^2$, $1 \le t \le N-1$. Generally, $y_t$ can not have the same value because

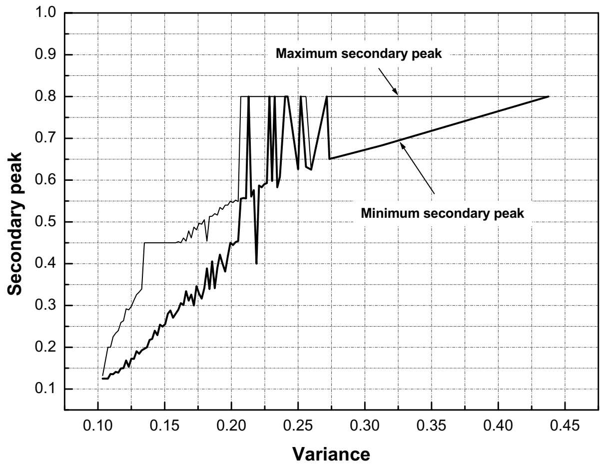


Figure 2: Relationship between the secondary peak and the variance of PRT sets when $N = 32$ and $W = 8$. All possible PRT sets are used.

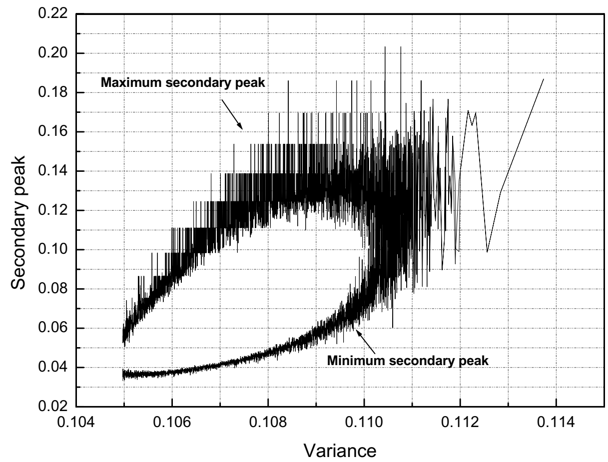


Figure 3: Relationship between the secondary peak and the variance for $N = 1024$ and $W = 102$. Randomly generated $10^7$ PRT sets are used.

it is generated by IFFT-ing the characteristic sequence of the PRT set in the TR scheme. Therefore, in this situation, it is not possible to prove that the minimization of the secondary peak is equivalent to the minimization of $\sigma_{\mathbf{y}}^2$. Thus, by doing numerical analysis, we investigate the relationship between the secondary peak and the variance $\sigma_y^2$ of the IFFT-ed signal $y_t$ of the PRT sets. In Figs. 2 and 3, the variance and the secondary peak are calculated for all the PRT sets when $N = 32$ and $W = 8$ and for $10^7$ randomly selected PRT sets when $N = 1024$ and $W = 102$, where the first peak power is normalized to 1. The numerical results show that the PRT sets with the same variance $\sigma^2$ can have the various secondary peaks and the secondary peak value statistically tends to decrease as $\sigma^2$ decreases. From the numerical results, it is worth mentioning that the PRT sets with the minimum secondary peak are contained in the PRT sets with the minimum or near minimum variance although the PRT set with the minimum variance does not guarantee the minimum secondary peak.

Now, we are going to evaluate the variance ${\sigma_{\mathbf{p}}}^2$ of $\mathbf{p}$. The power spectrum of $p_t$ [8] is expressed as

$$|p_t|^2 = \frac{1}{N}R_0 + \frac{1}{N}\sum_{\tau=1}^{N-1} R_\tau \cos(j2\pi\frac{\tau}{N}t) \tag{5}$$

where $R_\tau$ denotes the aperiodic autocorrelation of $\mathbf{P}$ defined by

$$R_\tau = \sum_{k=0}^{N-1-\tau} P_k P_{k+\tau}^* = \sum_{k=0}^{N-1-\tau} P_k P_{k+\tau}. \tag{6}$$

The time average $\mu$ of $|p_t|^2$ is given as

$$\mu = \frac{1}{N}\sum_{t=0}^{N-1}|p_t|^2 = \frac{1}{N}R_0. \tag{7}$$

Using (5) and (7), the variance $\sigma_{\mathbf{p}}^2$ of $|p_t|^2$ is obtained as

$${\sigma_{\mathbf{p}}}^2 = \frac{2}{N^2}\sum_{\tau=1}^{N-1}(R_\tau^2 + R_\tau R_{N-\tau}). \tag{8}$$

Table 1: Difference table of $(13, 4, 1)$ difference set $\mathbf{D} = \{2, 3, 5, 11\}$.

| mod 13 | 2 | 3 | 5 | 11 |
|---|---|---|---|---|
| 2 | 0 | 1 | 3 | 9 |
| 3 | 12 | 0 | 2 | 8 |
| 5 | 10 | 11 | 0 | 6 |
| 11 | 4 | 5 | 7 | 0 |

Since the variance of the time domain kernel $\mathbf{p}$ is expressed as the sum of the aperiodic autocorrelation of frequency domain kernel $\mathbf{P}$, the PRT set with low aperiodic autocorrelation has low variance and low secondary peak. Since the characteristic sequence of the cyclic difference set, which will be explained in the following has a good aperiodic autocorrelation property, the cyclic difference set could be a good PRT set.

A cyclic difference set, also called a cyclic $(v, q, \lambda)$ difference set, is a set $\mathbf{D} = \{d_0, d_1, \cdots, d_{q-1}\}$ of $q$ integers distinct modulo $v$ such that the congruence $d_i - d_j \equiv t \pmod v$ has exactly $\lambda$ solution pairs $(d_i, d_j)$ of elements of $\mathbf{D}$ for each integer $t$, $1 \le t \le v - 1$ [11].

Since there are $q(q-1)$ choices of $d_i \neq d_j$ from $\mathbf{D}$, giving $v - 1$ values of $t$ exactly $\lambda$ times, we have $q(q-1) = (v-1)\lambda$ as a necessary condition for the existence of a cyclic $(v, q, \lambda)$ difference set.

For example, $\mathbf{D} = \{2, 3, 5, 11\}$ is a $(13, 4, 1)$ difference set as seen in the Table 1.

If a constant $g$ is added to each element of a cyclic $(v, q, \lambda)$ difference set $\mathbf{D}$, the new set $\mathbf{D}' = \mathbf{D} + g = \{d_0 + g, d_1 + g, \cdots, d_{q-1}+g\}$ is again a $(v, q, \lambda)$ difference set, with the same difference table, because $(d_i+g)-(d_j+g) \equiv d_i - d_j \pmod v$. In this case, $\mathbf{D}$ and $\mathbf{D}'$ are often considered the same cyclic difference set.

It is proved that $m\mathbf{D} = \{md_0, md_1, \cdots, md_{q-1}\}$ becomes a cyclic $(v, q, \lambda)$ difference set again if $m$ is relative prime to $v$ such that $\gcd(v, m) = 1$. Combining these two operations,

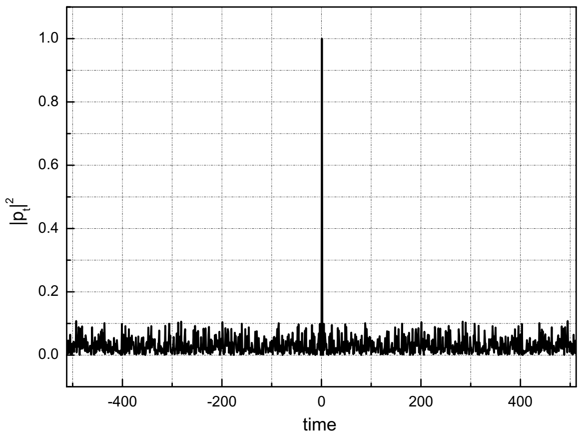


Figure 4: Power $|p_t|^2$ of the PRT set chosen from the randomly generated $10^7$ sets when $N = 1024$ and $W = 32$.

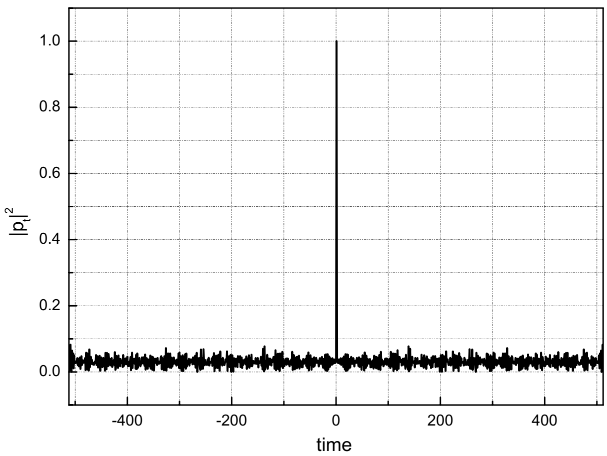


Figure 5: Power $|p_t|^2$ of the PRT set generated by using the difference set when $N = 1024$ and $W = 32$.

from a given cyclic $(v, q, \lambda)$ difference set $\mathbf{D}$, $v\phi(v)$ difference sets with the same parameters can be generated in the form $m\mathbf{D} + g$, where Euler phi-function $\phi(v)$ is the number of integers distinct modulo $v$ and relatively prime to $v$.

If $m\mathbf{D} = \mathbf{D} + g$ for some $g$, then $m$ is said to be a multiplier. If $m$ is a multiplier of the cyclic difference set $\mathbf{D}$, there is a translate $\mathbf{D}' = \mathbf{D} + a$ of $\mathbf{D}$ such that $m\mathbf{D}' = \mathbf{D}'$, provided that $\gcd(v, m-1) = 1$. It is proved that if $\mathbf{D}$ is a cyclic $(v, q, \lambda)$ difference set, and $p$ is a prime divisor of $k - \lambda$ with $p > \lambda$, then p is a multiplier of $\mathbf{D}$.

It is well known that cyclic difference sets can be constructed from pseudonoise (PN) sequences with ideal autocorrelation. Up to now, many works for cyclic difference sets with Singer parameters $(b^n - 1/b - 1, b^{n-1} - 1/b - 1, b^{n-2} - 1/b - 1)$ have been done on binary sequences, $b$-ary m-sequences, $b$-ary GMW sequences, and $b$-ary cascaded GMW sequences where $b$ is prime number.

Simulations are performed to show the availability of the difference set as the PRT set in the TR scheme,

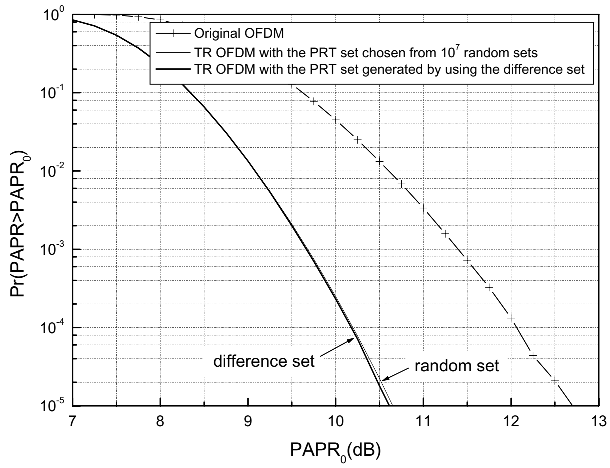


Figure 6: CCDF of the PAPR for the TR schemes with various PRT sets for $N = 1024$ and $W = 32$.

where the number of tones is 1024 and the number of reserved tones 32. Since there is no $(1024, 32, 1)$ difference set, it is needed to modify the existing $(1057, 33, 1)$ difference set $\mathbf{D} = \{0, 1, 31, 35, 40, 149, 155, 171, 191, 369, 396, 425, 450, 508, 521, 558, 613, 627, 651, 674, 700, 715, 717, 774, 777, 785, 795, 884, 912, 960, 1006, 1013, 1025\}$. The last element 1025 of the $(1057, 33, 1)$ difference set is removed to construct a PRT set for the above TR scheme.

$|p_t|^2$ of the PRT set chosen from the randomly generated $10^7$ sets is shown in Fig. 4 and $|p_t|^2$ of the proposed PRT set in Fig. 5. The secondary peak of the PRT set generated by using the difference set is smaller than that of the PRT set chosen from the randomly generated $10^7$ sets.

In Fig. 6, the PAPR reduction performance of the PRT set generated by using the cyclic difference set is compared with that of the PRT set chosen from the randomly generated $10^7$ sets in the TR scheme. The PAPR of the original OFDM signal without the PAPR reduction scheme is about 12.7dB at $10^{-5}$. The two sets reduce PAPR by about 2dB at $10^{-5}$ and the PAPR reduction performance of the PRT set generated by using the cyclic difference set is slightly better than that of the PRT set chosen from the randomly generated $10^7$ sets.

## IV. CONCLUSION

It is known that the optimal choice of the frequency domain kernel corresponds to selecting the time domain kernel such that the secondary peaks or sidelobes of $\mathbf{p}$ are minimized. But, it cannot be solved for practical values of $N$. Thus, we evaluate the relationship between the secondary peak and variance of the time domain kernel by the numerical analysis. The results show that the PRT sets with the same variance can have various secondary peaks and the secondary peak value statistically tends to decrease as the variance decreases. The PRT sets with the minimum secondary peak are contained in the PRT sets with the minimum or near minimum variance although the PRT set with the minimum variance does not guarantee the minimum secondary peak.

Since the variance of the time domain kernel is expressed as the sum of the aperiodic autocorrelation of frequency domain kernel, the PRT set with low aperiodic autocorrelation has low variance and low secondary peak. Since the characteristic sequence of the cyclic difference set has a good aperiodic autocorrelation property, the cyclic difference set could be a good PRT set. The near optimal PRT set of the TR scheme is constructed using the cyclic difference set and the PAPR reduction performance of the PRT set is compared with that of the PRT set chosen from the randomly generated $10^7$ sets. The PAPR reduction performance of the PRT set generated by using the cyclic difference set is slightly better than that of the PRT set chosen from the randomly generated $10^7$ sets.

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