

On the Diversity Analysis of Decode-and-Forward Protocol With Multiple Antennas

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Abstract—In this paper, a near maximum-likelihood (ML) decoder for orthogonal decode-and-forward (ODF) protocol with multiple antennas is proposed and the upper bound on the average pairwise error probability (PEP) for large signal to noise ratio is also derived, regardless of the modulation scheme. From the upper bound on the average PEP, we show that the ODF protocol with near ML decoder can achieve full diversity $M_S M_D + M_R \min(M_S, M_D)$ for a cooperative network consisting of one source, one relay, and one destination with M_S , M_R , and M_D antennas, respectively.

I. INTRODUCTION

Recently, the cooperative networks have been extensively studied [1], [2]. For the single antenna case, the maximum-likelihood (ML) decoder for decode-and-forward (DF) protocol has been presented only for binary phase shift keying (BPSK) and a suboptimal low-complexity decoder, called λ maximum ratio combining (λ -MRC), was proposed in [1]. In [2], a cooperative MRC (C-MRC) was proposed and it was proved that C-MRC for uncoded orthogonal DF (ODF) protocol can achieve full diversity where ‘orthogonal’ means that the source and the relay transmit signals through orthogonal channels. However, the above results are valid for the single antenna case and the ML decoder and the diversity for ODF protocol with multiple antennas have not been studied yet.

In this paper, we propose a near ML decoder for ODF protocol with multiple antennas and derive the upper bound on the average pairwise error probability (PEP) for large signal to noise ratio (SNR), regardless of the modulation scheme. From the upper bound on the average PEP, we show that the near ML decoder of the ODF protocol can achieve the maximum diversity $M_S M_D + M_R \min(M_S, M_D)$ for a cooperative network with one source, one relay, and one destination, where M_S , M_R , and M_D are the numbers of the antennas at the source, relay, and destination, respectively.

The following notations will be used in this paper: capital letter denotes a matrix; I_n denotes the $n \times n$ identity matrix; $\mathbb{C}^{n \times m}$ denotes a set of $n \times m$ complex matrices; $\|\cdot\|$ and $\text{tr}(\cdot)$ represent the Frobenius norm and the trace of a matrix, respectively; $E\{\cdot\}$ is the expectation; the superscript $(\cdot)^\dagger$ denotes the complex conjugate transpose; $\text{Re}(\cdot)$ means the real part of a complex number. For $A \in \mathbb{C}^{n \times m}$, $A \sim \mathcal{CN}(0, \sigma^2 I_{nm})$ denotes that the elements of A are independent and identically distrib-

uted (i.i.d.) circularly symmetric Gaussian random variables with zero mean and variance σ^2 .

II. A NEAR ML DECODER FOR ODF PROTOCOL

A. System Model

A cooperative ODF network with one source, one relay, and one destination using half duplex transmission is shown in Fig. 1. It is also assumed that the channels are frequency-flat quasi-static fading channels, the relay knows the channel state information (CSI) of source-relay (SR) channel, and the destination knows the CSI of SR, source-destination (SD), and relay-destination (RD) channels.

Let L be the number of transmitted data symbols through the first and second phases. Then, in the first phase, the source broadcasts $M_S \times T_1$ codeword $X_S(x)$ constructed from L data symbols $x = (x_1, x_2, \dots, x_L)$ to the relay and the destination and, in the second phase, the relay sends $M_R \times T_2$ codeword $X_R(x_R)$ constructed from L symbols $x_R = (x_1^R, x_2^R, \dots, x_L^R)$ which are the decoded symbols by the relay in the first phase. Then, the received signals at the relay and the destination in the first phase are

$$\begin{aligned} Y_{SR} &= KX_S(x) + N_{SR}, \\ Y_{SD} &= GX_S(x) + N_{SD}, \end{aligned} \quad (1)$$

respectively, where $K \in \mathbb{C}^{M_R \times M_S}$ and $G \in \mathbb{C}^{M_D \times M_S}$ are the channel coefficient matrices of the SR and SD channels, distributed as $K \sim \mathcal{CN}(0, \sigma_{SR}^2 I_{M_R M_S})$ and $G \sim \mathcal{CN}(0, \sigma_{SD}^2 I_{M_D M_S})$, respectively. $N_{SR} \in \mathbb{C}^{M_R \times T_1}$ and $N_{SD} \in \mathbb{C}^{M_D \times T_1}$ represent the noise matrices in the first phase at the relay and the destination with $N_{SR} \sim \mathcal{CN}(0, \sigma^2 I_{M_R T_1})$ and $N_{SD} \sim \mathcal{CN}(0, \sigma^2 I_{M_D T_1})$, respectively. In the second phase, the received signal at the destination is given as

$$Y_{RD} = FX_R(x_R) + N_{RD} \quad (2)$$

where $F \in \mathbb{C}^{M_D \times M_R}$ is the channel coefficient matrix of the RD channel, distributed as $F \sim \mathcal{CN}(0, \sigma_{RD}^2 I_{M_D M_R})$, and $N_{RD} \in \mathbb{C}^{M_D \times T_2}$ is the noise matrix at the destination with $N_{RD} \sim \mathcal{CN}(0, \sigma^2 I_{M_D T_2})$. We assume that the total average transmit power of transmit antennas is 1 at each of source and relay, and therefore, the average transmit SNR is $\rho = \frac{1}{\sigma^2}$.

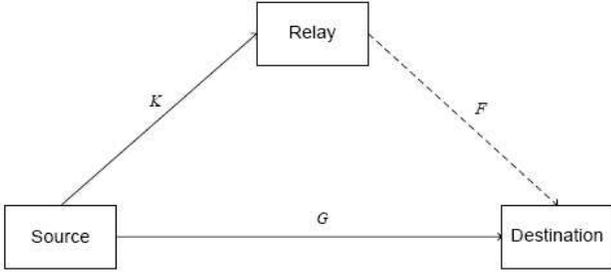


Fig. 1. The ODF protocol using one relay. The solid line denotes the first phase transmission and the dashed line denotes the second phase transmission.

B. A Near ML Decoder

For the ODF protocol, the destination does not know what the relay transmits and, thus, the ML decoder can be written as

$$\begin{aligned}
 \hat{x} &= \arg \max_{x \in \mathcal{A}^L} p(Y_{SD}, Y_{RD} | x) \\
 &= \arg \max_{x \in \mathcal{A}^L} \sum_{\hat{x}_R \in \mathcal{A}^L} p(Y_{SD}, Y_{RD} | x, \hat{x}_R) P_{SR}(\hat{x}_R | x) \\
 &= \arg \max_{x \in \mathcal{A}^L} p(Y_{SD} | X_S(x)) \sum_{\hat{x}_R \in \mathcal{A}^L} p(Y_{RD} | X_R(\hat{x}_R)) P_{SR}(\hat{x}_R | x) \\
 &= \arg \max_{x \in \mathcal{A}^L} \left[-\frac{\|Y_{SD} - GX_S(x)\|^2}{\sigma^2} \right. \\
 &\quad \left. + \ln \sum_{\hat{x}_R \in \mathcal{A}^L} \exp\left(\frac{-\|Y_{RD} - FX_R(\hat{x}_R)\|^2 + \sigma^2 \ln P_{SR}(\hat{x}_R | x)}{\sigma^2}\right) \right] \quad (3)
 \end{aligned}$$

where \mathcal{A} is the signal set for the M -ary signal constellation and $P_{SR}(\hat{x}_R | x)$ is the probability that the relay decodes the received signal to \hat{x}_R when the source transmits the signal x in the first phase.

It is easy to see that if the codewords X_S and X_R are single-symbol decodable for multiple-antenna communication system, a cooperative ODF protocol with multiple antennas is also single-symbol decodable.

Since $P_{SR}(\hat{x}_R | x)$ is very difficult to derive for codeword X_S , instead of $P_{SR}(\hat{x}_R | x)$ in (3), we will use the PEP of deciding \hat{x}_R at the relay when x is transmitted from the source, i.e., $P_{SR}(x \rightarrow \hat{x}_R)$, which is derived in [3]. Although the PEP is not equal to $P_{SR}(\hat{x}_R | x)$, it can be a good substitution for $P_{SR}(\hat{x}_R | x)$ to find the solution of (3). For large SNR, i.e., $\sigma^2 \rightarrow 0$, the approximation $\ln \sum_i e^{x_i} \approx \max_i x_i$ is also used. Finally, the ML decoder can be approximated to

$$\begin{aligned}
 \hat{x} &= \arg \min_{x \in \mathcal{A}^L} \left\{ \|Y_{SD} - GX_S(x)\|^2 \right. \\
 &\quad \left. + \min_{\hat{x}_R \in \mathcal{A}^L} [\|Y_{RD} - FX_R(\hat{x}_R)\|^2 - \sigma^2 \ln P_{SR}(x \rightarrow \hat{x}_R)] \right\}. \quad (4)
 \end{aligned}$$

We call this decoder a near ML decoder. In the next section, we will analyze the error performance of the near ML decoder. Note that the performance of near ML decoder approaches to that of ML decoder as the SNR increases.

III. DIVERSITY ANALYSIS OF ODF PROTOCOL USING PEP

The following theorem is used to derive the PEP for near ML decoder.

Theorem 1: Let A and B be complex matrices satisfying $\|B\|^2 > \|A\|^2$ and N be the matrix with the i.i.d. complex Gaussian entries with distribution $\mathcal{CN}(0, \sigma^2)$. Then, for $\sigma^2 \rightarrow 0$, $\|B + N\|^2 \geq \|A + N\|^2$ in probability, i.e.,

$$\lim_{\sigma^2 \rightarrow 0} P(\|B + N\|^2 \geq \|A + N\|^2) = 1.$$

Proof:

$$\begin{aligned}
 &\lim_{\sigma^2 \rightarrow 0} P(\|B + N\|^2 \geq \|A + N\|^2) \\
 &= \lim_{\sigma^2 \rightarrow 0} P(2\text{Re}\{\text{tr}((B - A)N^\dagger)\} \geq -(\|B\|^2 - \|A\|^2)) \\
 &\stackrel{(a)}{=} \lim_{\sigma^2 \rightarrow 0} Q - \frac{\|B\|^2 - \|A\|^2}{\sqrt{2\sigma^2\|B - A\|^2}} = 1
 \end{aligned}$$

where $Q(x) = \frac{1}{\sqrt{2\pi}} \int_x^\infty e^{-\frac{y^2}{2}} dy$ and the equality (a) holds because the variance of Gaussian random variable $2\text{Re}\{\text{tr}((B - A)N^\dagger)\}$ is $2\sigma^2\|B - A\|^2$ as shown in [3]. \square

Since the relay can transmit any symbol, the average PEP should be written as

$$E_{K,G,F}[P(x \rightarrow \tilde{x})] = \sum_{x_R} E_{K,G,F}[P(x \rightarrow \tilde{x} | x_R) P_{SR}(x_R | x)] \quad (5)$$

where $P(x \rightarrow \tilde{x} | x_R)$ denotes the conditional PEP of deciding \tilde{x} at the destination when x and x_R are transmitted from the source and relay, respectively. We assume that the source transmits codeword $X_S(x)$ in the first phase and the relay decodes the received signal to x_R , constructs the codeword $X_R(x_R)$, and transmits it to the destination in the second phase. Then, the conditional PEP in (5) can be written as

$$\begin{aligned}
 &P(x \rightarrow \tilde{x} | x_R) \\
 &= P(m([Y_{SD}, Y_{RD}], x | x, x_R) > m([Y_{SD}, Y_{RD}], \tilde{x} | x, x_R)), \quad (6)
 \end{aligned}$$

where $m([Y_{SD}, Y_{RD}], x | x, x_R)$ and $m([Y_{SD}, Y_{RD}], \tilde{x} | x, x_R)$ are the metrics in (4) of deciding x and \tilde{x} for the given x and x_R transmitted from the source and the relay, respectively. The PEP $P_{SR}(x \rightarrow z)$ for given K in (4), as shown in [3], is

$$P_{SR}(x \rightarrow z) = Q\left(\sqrt{\frac{\|K(X_S(x) - X_S(z))\|^2}{2\sigma^2}}\right).$$

If $z = x$, $\lim_{\sigma^2 \rightarrow 0} \sigma^2 \ln P_{SR}(x \rightarrow z) = 0$ and otherwise,

$$\begin{aligned}
 &\lim_{\sigma^2 \rightarrow 0} \sigma^2 \ln P_{SR}(x \rightarrow z) \\
 &= \lim_{\sigma^2 \rightarrow 0} \sigma^2 \ln Q \sqrt{\frac{\|K(X_S(x) - X_S(z))\|^2}{2\sigma^2}} \\
 &= \lim_{\sigma^2 \rightarrow 0} \sigma^2 \ln \frac{e^{-\frac{\|K(X_S(x) - X_S(z))\|^2}{4\sigma^2}}}{\sqrt{2\pi \frac{\|K(X_S(x) - X_S(z))\|^2}{2\sigma^2}}} \\
 &= -\frac{\|K(X_S(x) - X_S(z))\|^2}{4}.
 \end{aligned}$$

Therefore, for large SNR, the metrics in (6) can be rewritten as (7) and (8).

$$m([Y_{SD}, Y_{RD}], \mathbf{x} | \mathbf{x}, \mathbf{x}_R) = \|N_{SD}\|^2 + \min_{\hat{\mathbf{x}}_R} [\|F(X_R(\mathbf{x}_R) - X_R(\hat{\mathbf{x}}_R)) + N_{RD}\|^2 + \frac{1}{4} \|K(X_S(\mathbf{x}) - X_S(\hat{\mathbf{x}}_R))\|^2] \quad (7)$$

$$m([Y_{SD}, Y_{RD}], \tilde{\mathbf{x}} | \mathbf{x}, \mathbf{x}_R) = \|G(X_S(\mathbf{x}) - X_S(\tilde{\mathbf{x}})) + N_{SD}\|^2 + \min_{\hat{\mathbf{x}}_R} [\|F(X_R(\mathbf{x}_R) - X_R(\hat{\mathbf{x}}_R)) + N_{RD}\|^2 + \frac{1}{4} \|K(X_S(\tilde{\mathbf{x}}) - X_S(\hat{\mathbf{x}}_R))\|^2] \quad (8)$$

$$m([Y_{SD}, Y_{RD}], \mathbf{x} | \mathbf{x}, \mathbf{x}_R) \leq \|N_{SD}\|^2 + \|N_{RD}\|^2 + \frac{1}{4} \|K(X_S(\mathbf{x}) - X_S(\mathbf{x}_R))\|^2 \quad (9)$$

$$m([Y_{SD}, Y_{RD}], \tilde{\mathbf{x}} | \mathbf{x}, \mathbf{x}_R) \geq \|G(X_S(\mathbf{x}) - X_S(\tilde{\mathbf{x}})) + N_{SD}\|^2 + \|N_{RD}\|^2 \quad (10)$$

$$\begin{aligned} & E_{K,G,F} [P(\mathbf{x} \rightarrow \tilde{\mathbf{x}} | \mathbf{x}_R) P_{SR}(\mathbf{x}_R | \mathbf{x})] \\ & \leq E_{K,G,F} \left[P(\|N_{SD}\|^2 + \|N_{RD}\|^2 + \frac{1}{4} \|K(X_S(\mathbf{x}) - X_S(\mathbf{x}_R))\|^2 > \|G(X_S(\mathbf{x}) - X_S(\tilde{\mathbf{x}})) + N_{SD}\|^2 + \|N_{RD}\|^2) \right. \\ & \quad \left. e^{-\frac{\|K(X_S(\mathbf{x}) - X_S(\mathbf{x}_R))\|^2}{4\sigma^2}} \right] \end{aligned} \quad (12)$$

$$\begin{aligned} m([Y_{SD}, Y_{RD}], \tilde{\mathbf{x}} | \mathbf{x}, \mathbf{x}_R) & \geq \|G(X_S(\mathbf{x}) - X_S(\tilde{\mathbf{x}})) + N_{SD}\|^2 \\ & + \min [\|N_{RD}\|^2 + \frac{1}{4} \|K(X_S(\tilde{\mathbf{x}}) - X_S(\mathbf{x}))\|^2, \|F(X_R(\mathbf{x}) - X_R(\hat{\mathbf{x}}_R^{\min})) + N_{RD}\|^2] \end{aligned} \quad (13)$$

$$\begin{aligned} E_{K,G,F} [P(\mathbf{x} \rightarrow \tilde{\mathbf{x}} | \mathbf{x}) P_{SR}(\mathbf{x} | \mathbf{x})] & \leq E_{K,G,F} \left[P(\|N_{SD}\|^2 + \|N_{RD}\|^2 > \|G(X_S(\mathbf{x}) - X_S(\tilde{\mathbf{x}})) + N_{SD}\|^2 \right. \\ & \quad \left. + \min [\|N_{RD}\|^2 + \frac{1}{4} \|K(X_S(\tilde{\mathbf{x}}) - X_S(\mathbf{x}))\|^2, \|F(X_R(\mathbf{x}) - X_R(\hat{\mathbf{x}}_R^{\min})) + N_{RD}\|^2]) \right] \end{aligned} \quad (14)$$

Next, we will consider two cases, $\mathbf{x}_R \neq \mathbf{x}$ and $\mathbf{x}_R = \mathbf{x}$.

Case i) $\mathbf{x}_R \neq \mathbf{x}$: By considering only the case $\hat{\mathbf{x}}_R = \mathbf{x}_R$ in the min function of (7), the upper bound on $m([Y_{SD}, Y_{RD}], \mathbf{x} | \mathbf{x}, \mathbf{x}_R)$ is obtained as (9). For large SNR, by Theorem 1, the lower bound on $m([Y_{SD}, Y_{RD}], \tilde{\mathbf{x}} | \mathbf{x}, \mathbf{x}_R)$ is given as (10). Since $P_{SR}(\mathbf{x}_R | \mathbf{x})$ in (5) is less than the PEP $P_{SR}(\mathbf{x} \rightarrow \mathbf{x}_R)$, by using the PEP and $Q(x) \leq e^{-\frac{x^2}{2}}$ for $x > 0$, we also have

$$\begin{aligned} P_{SR}(\mathbf{x}_R | \mathbf{x}) & \leq Q\left(\sqrt{\frac{\|K(X_S(\mathbf{x}) - X_S(\mathbf{x}_R))\|^2}{2\sigma^2}}\right) \\ & \leq e^{-\frac{\|K(X_S(\mathbf{x}) - X_S(\mathbf{x}_R))\|^2}{4\sigma^2}}. \end{aligned} \quad (11)$$

Thus, for $\mathbf{x}_R \neq \mathbf{x}$, by using (9), (10), and (11), the summand in (5) can be upper bounded as (12).

Case ii) $\mathbf{x}_R = \mathbf{x}$: For large SNR, by Theorem 1, we have

$$m([Y_{SD}, Y_{RD}], \mathbf{x} | \mathbf{x}, \mathbf{x}_R) = \|N_{SD}\|^2 + \|N_{RD}\|^2.$$

For $m([Y_{SD}, Y_{RD}], \tilde{\mathbf{x}} | \mathbf{x}, \mathbf{x}_R)$, we consider two cases, $\hat{\mathbf{x}}_R = \mathbf{x}$ and $\hat{\mathbf{x}}_R \neq \mathbf{x}$. Then, from Theorem 1, we have (13), where $\hat{\mathbf{x}}_R^{\min} = \arg \min_{\hat{\mathbf{x}}_R \neq \mathbf{x}} \|F(X_R(\mathbf{x}) - X_R(\hat{\mathbf{x}}_R))\|^2$. Since $P_{SR}(\mathbf{x} | \mathbf{x}) \leq 1$, similarly to Case i), we can have (14).

Next, we will derive the diversity of ODF protocol with multiple antennas using the upper bound on the average PEP.

Theorem 2: For the ODF protocol with multiple antennas, the PEP of near ML decoding between symbol sets \mathbf{x} and $\tilde{\mathbf{x}}$ on the arbitrary constellation can be upper bounded as

$$\begin{aligned} & E_{K,G,F} [P(\mathbf{x} \rightarrow \tilde{\mathbf{x}})] \\ & \leq \left(\frac{4\sigma^2}{\omega_{\min} \sigma_{SD}^2}\right)^{r_S M_D} \left[2 \left(\frac{4\sigma^2}{\mu_{\min} \sigma_{RD}^2}\right)^{r_R M_D} + M^L \left(\frac{8\sigma^2}{\omega_{\min} \sigma_{SR}^2}\right)^{r_S M_R} \right] \end{aligned} \quad (15)$$

where ω_{\min} and μ_{\min} are the minimum values among nonzero eigenvalues of $(X_S(\mathbf{x}) - X_S(\mathbf{z}))(X_S(\mathbf{x}) - X_S(\mathbf{z}))^\dagger$ and $(X_R(\mathbf{x}) - X_R(\mathbf{z}))(X_R(\mathbf{x}) - X_R(\mathbf{z}))^\dagger$ of ranks r_S and r_R for all $\mathbf{z} \neq \mathbf{x}$, respectively. Therefore, the achievable diversity of the near ML decoding is $r_S M_D + \min(r_S M_R, r_R M_D)$ for any modulation scheme. The full diversity $M_S M_D + M_R \min(M_S, M_D)$ is achieved when $r_S = M_S$ and $r_R = M_R$.

Proof: This theorem can be proved through some manipulation using (12) and (14). We will omit the process.

Note that if single antenna is used at each of source, relay, and destination, the ODF protocol achieves the diversity 2.

IV. NUMERICAL RESULTS

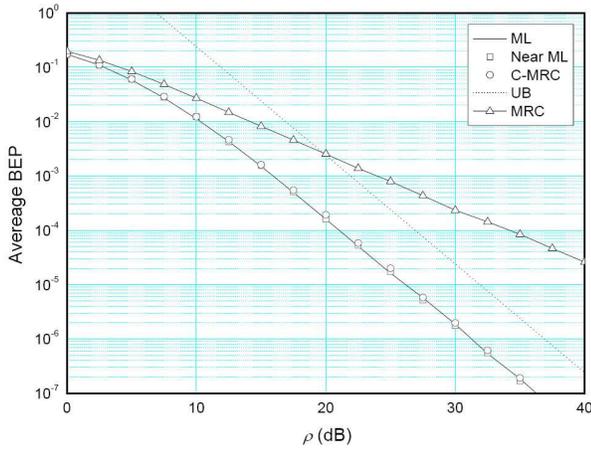


Fig. 2. Average BEP comparison of various decoders for ODF protocol with single antenna when QPSK is used.

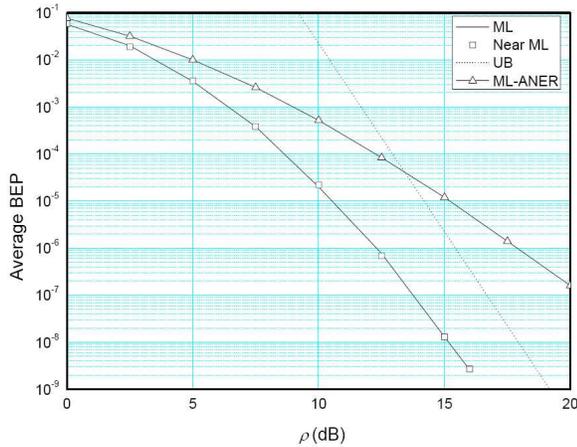


Fig. 3. Average BEP comparison of various decoders for ODF protocol with $M_S = M_R = M_D = 2$ when QPSK and Alamouti code are used at the source and relay.

For the ODF protocol with single antenna, Fig. 2 compares the average bit error probabilities (BEPs) of near ML, ML, C-MRC, and MRC decoders, and the upper bound on the average BEP for quadrature phase shift keying (QPSK). The ‘UB’ denotes the upper bound on the average BEP derived from the average PEP in (15). Fig. 2 shows that the near ML decoder has almost the identical performance as ML decoder, a little better than C-MRC, and better than MRC. The slope of the upper bound is almost the same as that of near ML performance in high SNR range. This shows that the near ML decoder can achieve the same diversity 2 as ML and C-MRC decoders which is larger than the diversity of MRC.

In Fig. 3, the average BEPs of ML, near ML, and ML assuming no error at the relay (ML-ANER) decoders are compared with the upper bound on the average BEP derived from (15) for $M_S = M_R = M_D = 2$ when QPSK and

Alamouti code are used at the source and relay. For multiple-antenna, MRC and C-MRC cannot be applied in general. Fig. 3 shows that the near ML decoder shows similar performance as ML decoder and better performance than ML-ANER decoder. The slope of the upper bound on the average BEP is similar to the average BEPs of ML and near ML decoders and larger than that of ML-ANER decoder for large SNR.

V. CONCLUSION

In this paper, a near ML decoder for ODF protocol with multiple antennas is proposed and the upper bound on the average PEP for large SNR is also derived, which is valid for any modulation scheme. Using the upper bound on the average PEP, we show that the ODF protocol using one source, one relay, and one destination with near ML can achieve full diversity $M_S M_D + M_R \min(M_S, M_D)$.

ACKNOWLEDGMENT

This work was supported by the IT R&D program of MKE/IITA. [2008-F-007-02, Intelligent Wireless Communication Systems in 3 Dimensional Environment]

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