New Performance Measure for Compressive Sensing Matrices Using k-Set Correlation

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***Abstract*—In this paper, we propose l_2-norm of k-set correlation as a new performance measure for compressive sensing matrices. We show that this measure is highly related to restricted isometry property. And we also demonstrate by simulation that it works well as a performance measure of sensing matrices.**

I. INTRODUCTION

One of the important issues with compressive sensing is design of the sensing matrix Φ. Restricted isometry property (RIP)[1] is the most well-known measure about performance of sensing matrices. If a sensing matrix Φ satisfies

$$(1-\delta_k)\,||x||^2 \le ||\Phi x||^2 \le (1+\delta_k)\,||x||^2 \tag{1}$$

for all possible k-sparse message vector x with isometry constant δ_k, then it is said that Φ obeys k-RIP.

It was proved in [1] that if Φ satisfies $2k$-RIP with suitable constant δ_{2k}, then perfect recovery of k-sparse vector x is possible. And Gaussian (or Bernoulli) random matrices with some constraints guarantee RIP with high probability[1]. Thus random matrices have been widely used as sensing matrices. But it is very hard to check whether a random matrix satisfies RIP or not. Furthermore, RIP is too strict condition.

Calderbank et al. proposed deterministic sensing matrix design with Statistical RIP (StRIP)[2]. StRIP is a statistical version of RIP. It has less strict conditions than RIP and can be easily checked for matrices with some constraints. Other design methods for deterministic sensing matrices are proposed, including matrix design from sequences[3].

In this paper, we will introduce a new performance measure for sensing matrices using k-set correlation vector which consists of coherences, i.e. correlations between two columns of a sensing matrix. It is possible to compare the performance of sensing matrices by using this measure.

II. NEW MEASURE FOR THE PERFORMANCE OF COMPRESSIVE SENSING MATRICES

Before introducing a new measure for sensing matrix, we will define the k-set correlation vector. k-set correlation of sensing matrix is a $\binom{k}{2}$-tuple vector. Each component of the vector is correlation between pair of columns in the set of k columns in the sensing matrix. Thus there are $\binom{N}{k}$ distinct k-set correlation vectors for a $M \times N$ sensing matrix.

The measure that we suggest is the distribution of l_2-norm of k-set correlation. l_2-norm of k-set correlation is highly related with restricted isometry constant δ_k in (1). Equation (1) can be represented as

$$\left|\,||\Phi x||^2 - ||x||^2\right| \le \delta_k\,||x||^2\,. \tag{2}$$

Let ϕ_i be an i-th column vector of sensing matrix Φ, and $r(i)$ be an i-th component of column vector r. Assume that ϕ_i's $(i = 1, 2, \cdots, N)$ have unit norm. Then it follows that

$$||\Phi x||^2 = \sum_{i=1}^{M}\left[\left(\sum_{j=1}^{N}\phi_j(i)x(j)\right)\overline{\left(\sum_{h=1}^{N}\phi_h(i)x(h)\right)}\right] \tag{3}$$

$$= ||x||^2 + \sum_{i=1}^{M}\sum_{j,h=1,j\neq h}^{N} x(j)\overline{x(h)}\phi_j(i)\overline{\phi_h(i)}. \tag{4}$$

Note that there are at most k nonzero elements in vector x. Without loss of generality, we could assume that $x(1), x(2), \cdots, x(k) \neq 0$ and others are zero. Then the summation $\sum_{j,h=1,j\neq h}^{N}$ in (4) is reduced to $\sum_{j,h=1,j\neq h}^{k}$. And we will define $\sum_{j,h} := \sum_{j,h=1,j\neq h}^{k}$ for simplicity. Let $x(j)\overline{x(h)} = x_{j,h}$ and $\sum_{i=1}^{M}\phi_j(i)\overline{\phi_h(i)} = c_{j,h}$. Then we can derive the following equation from (2) and (4).

$$\left|\sum_{j,h} x_{j,h}c_{j,h}\right| \le \delta_k\,||x||^2 \tag{5}$$

And it is possible to obtain the inequality for the left side of (5) from Cauchy-Schwartz inequality.

$$\left|\sum_{j,h} x_{j,h}c_{j,h}\right| \le \sqrt{\left(\sum_{j,h}|x_{j,h}|^2\right)\left(\sum_{j,h}|c_{j,h}|^2\right)} \tag{6}$$

Since

$$\sum_{j,h}|x_{j,h}|^2 = \sum_{j=1}^{k}\sum_{h=1}^{k}\left|x(j)\overline{x(h)}\right|^2 - \sum_{j=1}^{k}|x(j)|^4 \tag{7}$$

$$= ||x||^4 - \sum_{j=1}^{k}|x(j)|^4 \tag{8}$$

$$\le \frac{k-1}{k}\,||x||^4 \tag{9}$$

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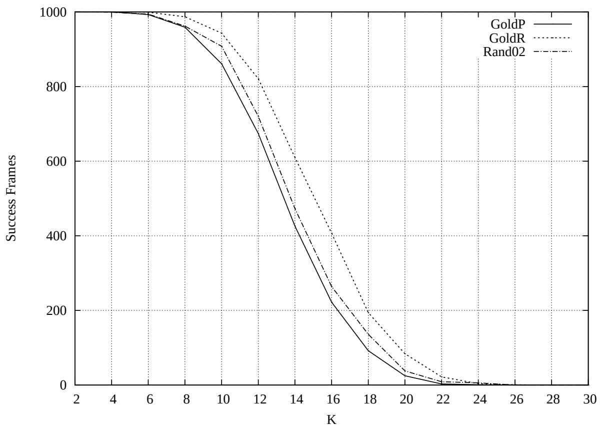

Fig. 1. Reconstruction performance with OMP

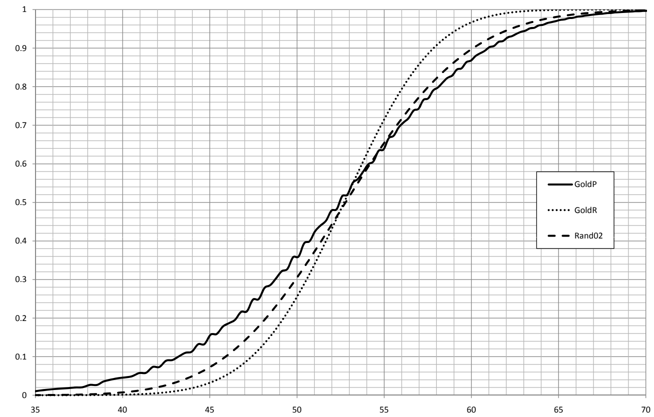

Fig. 2. Cumulative distribution of l_2-norm of 10-set correlation

and

$$\sum_{j,h} |c_{j,h}|^2 = \sum_{j,h} \left(\frac{\left|c'_{j,h}\right|}{M} \right)^2 = \frac{1}{M^2} \sum_{j,h} \left|c'_{j,h}\right|^2, \qquad (10)$$

$$\left| \sum_{j,h} x_{j,h} c_{j,h} \right| \leq \frac{1}{M} \sqrt{\frac{k-1}{k} \sum_{j,h} \left|c'_{j,h}\right|^2} \cdot ||x||^2 \qquad (11)$$

is derived from (6). Equation (9) is obtained from Cauchy-Schwartz inequality and $c'_{j,h}$ denotes the scaled correlation for the case of $||\phi_i||^2 = M$.

Restricted isometry constant δ_k is a minimum number that guarantees (5) for all k-sparse vector x. From this and (11),

$$\delta_k \leq \max_{x, ||x||_0 \leq k} \frac{1}{M} \sqrt{\frac{k-1}{k} \sum_{j,h} \left|c'_{j,h}\right|^2}. \qquad (12)$$

If Φ and k are given, the right side of (12) could be represented as $\max_{x,||x||_0 \leq k} \left(c \cdot \sqrt{\sum_{j,h} \left|c'_{j,h}\right|^2} \right)$ for constant c. Since $\sum_{j,h} \left|c'_{j,h}\right|^2 = 2 \sum_{j,h=1,j>h}^{k} \left|c'_{j,h}\right|^2$, this term is proportional to the maximal l_2-norm of k-set correlation for fixed Φ and k. In other words, δ_k is upper bounded by constant multiple of maximal l_2-norm of k-set correlation.

To satisfy RIP, we must reduce the maximal l_2-norm of k-set correlation. But our purpose is not to satisfy RIP but to find a performance measure for sensing matrix in average sense. Thus we will focus on the distribution of l_2-norm of k-set correlation.

III. Simulation Results

Fig. 1 shows the number of successful recovery per 1000 frames for several sensing matrices. Message vector is k-sparse and its nonzero components are random complex number with uniform magnitude and phase. Orthogonal matching pursuit (OMP)[4] is used for reconstruction method. All the matrices used for simulations are 63×4096 matrices with binary components ± 1. GoldP (GoldR) represents the matrix constructed by Gold sequence[5] with preferred (reciprocal) pair. Rand02 represents binary random matrix from Bernoulli distribution with $p = 0.5$. It is shown in Fig. 1 that the order of reconstruction performance with OMP is GoldR, Rand02, GoldP.

Fig. 2 shows the cumulative distribution of l_2-norm of 10-set correlation. We draw only the case of $k = 10$, but we confirm that the graphs have very similar tendency for the case of $k = 2, 3, \cdots, 20$. The higher parts of l_2-norm dominate the performance of sensing matrix. Thus we could estimate from Fig. 2 that the order of performance is GoldR, Rand02, GoldP. It is exactly identical with the order in Fig. 1.

We only present the cases of binary matrices in this paper, but it can be extended to general complex matrices. Unfortunately, distribution of l_2-norm of k-set correlation is also hard to compute for large N and k. Simple computing methods for the distribution are left as a further work.

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