Protograph Design for QC LDPC Codes With Large Girth

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***Abstract*—In this paper, all subgraph patterns of protographs which prevent quasi-cyclic (QC) low-density parity-check (LDPC) codes from having large girth are searched in allowance with multiple edges based on graph theoretic approach. A systematic construction of protograph with multiple edges using combinatorial design is proposed for QC LDPC codes with girth larger than or equal to 12.**

I. Introduction

Low-density parity-check (LDPC) codes [1] have been one of the major topics for many coding theorists over the past decade due to its near capacity-approaching performance. Since the low decoding complexity for the LDPC codes is achieved by various iterative decoding algorithms, LDPC codes can be adopted in many practical applications. Especially quasi-cyclic (QC) LDPC codes are well suited for hardware implementation composed of simple shift registers due to the regularity in their parity-check matrices.

Thorpe [2] introduced the concept of *protograph codes*, a class of LDPC codes lifted from protographs. QC LDPC codes belong to protograph codes because they can be regarded as the lifted ones from the corresponding protographs using cyclic permutation. Therefore, the construction of good QC LDPC codes mainly depends on successful design of their protographs.

There have been many efforts on the construction for QC LDPC codes with large girth. In [3], they constructed some protographs whose protograph codes can have large girth using combinatorial design, and considered only the protographs which have a single edge between two nodes. However, the QC LDPC codes lifted from the protographs with multiple edges between two nodes can show better performance due to the flexibility in adjusting their degree distribution, and they can have larger minimum distance according to [4]. In this paper, we suggest the method of protograph design for QC LDPC codes with multiple edges and large girth using combinatorial design.

II. Subgraph Patterns of Protographs

In order for a QC LDPC code to have large girth, its protograph should not have any subgraph pattern which prevents the LDPC code from having large girth. For search of those subgraph patterns, we need to review some remarkable results based on graph theory [5].

A. Preliminary

By abuse of notation, let P denote both a (bipartite) protograph and its incidence matrix. And let $P=[p_{ij}]$, where p_{ij} is a non-negative integer, and thus the horizontal node i and the vertical node j is connected each other via p_{ij} edge(s). If $p_{ij} \geq 2$, there are multiple edges between the two nodes. As the work of [5], we will define two classes of graphs. Define an (a_1,a_2,a_3)-theta graph, denoted by $T(a_1,a_2,a_3)$, to be a graph consisting of two vertices, each of degree three, that are connected to each other via three disjoint paths A_1, A_2, A_3 of the number of edges $a_1 \geq 1$, $a_2 \geq 1$, and $a_3 \geq 1$, respectively. And also define a $(c_1,c_2;b)$-dumbbell graph, denoted $D(c_1,c_2;b)$ to be a connected graph consisting of two edge-disjoint cycles C_1 and C_2 of the number of edges $c_1 \geq 1$ and $c_2 \geq 1$, respectively, that are connected by a path B of the number of edges $b \geq 0$.

The girth of a QC LDPC code is determined by both the structure of the protograph and its shift values. However, we can derive the upper bound of the girth by introducing the concept of *inevitable cycle* [3] for the protograph. The length of an inevitable cycle for a protograph P is defined as the maximum of positive integer n such that for every lift size and every shift value assignment, the QC LDPC code lifted from P must have a cycle of length n.

B. Searching Subgraph Patterns

The subgraph pattern P_{2i} is defined as follows: 1) P_{2i} has the inevitable cycle with length $2i$; 2) P_{2i} does not have any subgraph which has an inevitable cycle with the length smaller than $2i$; 3) The number of rows is not smaller than that of columns; 4) For an isomorphic class of graphs, only one matrix P_{2i} must be given as a representative. The next two theorems can be directly derived from [5], and their proofs are omitted.

Theorem 1: P_{2i} must be either a theta graph or a dumbbell graph.

Theorem 2: $T(a_1,a_2,a_3)$ has an inevitable cycle with the length of $2(a_1+a_2+a_3)$. $D(c_1,c_2;b)$ has an inevitable cycle with the length of $2(c_1+c_2)+4b$.

Now we can find all subgraph patterns from $T(a_1, a_2, a_3)$ and $D(c_1, c_2; b)$. Since P_{2i}'s are bipartite graphs, the parameters should satisfy the following conditions: 1) $a_1 \geq a_2 \geq a_3 \geq 1$; 2) a_1, a_2, a_3 have the same parity; 3) $c_1 \geq c_2 \geq 2$, $b \geq 0$; 4) c_1 and c_2 are even. All subgraph patterns with the length of the inevitable cycle up to 18 are listed as belows.

$$P_6 = [3]$$

$$P_8 = \begin{bmatrix} 2 & 2 \end{bmatrix}$$

$$P_{10} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$$

$$P_{12} = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}, \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

$$P_{14} = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

$$P_{16} = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 1 & 2 \end{bmatrix}, \begin{bmatrix} 2 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}, \begin{bmatrix} 2 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

$$P_{18} = \begin{bmatrix} 2 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$

III. Regular Protograph Construction

In this section, we will focus on the construction of regular protographs using combinatorial design.

A. Balanced Ternary Design

A *balanced ternary design* $BTD(V, B; \rho_1, \rho_2, R; K, \Lambda)$ [6] is an arrangement of V elements into B multisets, or blocks, each of cardinality K ($K \leq V$), satisfying: 1) Each element appears $R = \rho_1 + 2\rho_2$ times altogether, with multiplicity one in exactly ρ_1 blocks, with multiplicity two in exactly ρ_2 blocks; 2) Every pair of distinct elements appears Λ times; i.e., if m_{vb} is the multiplicity of the vth element in the bth block, then for every pair of distinct elements v and w, we have $\sum_{b=1}^{B} m_{vb} m_{wb} = \Lambda$.

B. Regular QC LDPC Codes With Girth Larger Than or Equal to 12

In order for a QC LDPC code to have the girth larger than or equal to 12, the protograph should not have P_6, P_8, P_{10} as its subgraphs. For P_6, '3' must not appear in the protograph. For P_8, any pair of '2's should not exist in a row and in a column. It is not simple for P_{10}, but we can construct the protograph which does not contain all of P_6, P_8, P_{10} by using balanced ternary design.

The incidence matrix of a $BTD(V, B; \rho_1, \rho_2, R; K, \Lambda)$ can be regarded as a $V \times B$ protograph. And the variable node degree and the check node degree of its lifted QC LDPC code become K and R, respectively. The condition $\rho_2 = 1$ means that only one '2' exists in each row of the protograph. And $\Lambda = 2$ implies that each column of the protograph may have at most one '2' and that P_{10} must not appear in the protograph. Table I lists all possible parameters of balanced ternary design with $R \leq 15$ [7] suitable for the protographs of QC LDPC codes. As an example, the incidence matrix of a $BTD(6, 12; 4, 1, 6; 3, 2)$ is shown as below, and we can check that no subgraph pattern P_{2i} with $i \leq 12$ does not appear.

TABLE I
Balanced Ternary Design with $\rho_2 = 1, \Lambda = 2, V/B < 1, R \leq 15$

V	6	12	9	20	12	30	42	48	42	15	60
B	12	24	27	40	48	60	63	64	84	75	100
K	3	4	3	5	3	6	8	9	7	3	9
R	6	8	9	10	12	12	12	12	14	15	15

$$\begin{bmatrix} 2 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 2 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 2 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 2 & 0 & 0 & 1 & 1 & 0 & 1 & 1 \end{bmatrix}$$

IV. Conclusion

The subgraph patterns of protographs which cause inevitable cycles were fully searched from the graph theoretic approach in allowance with multiple edges in the protographs. For QC LDPC codes with girth larger than or equal to 12, we proposed the combinatorial method of constructing protographs using balanced ternary design.

Acknowledgment

This work was supported by the National Research Foundation of Korea (NRF) grant funded by the Korea government (MEST) (No.2010-0000867), and also supported by the the IT R&D program of MKE/KEIT. [KI001809, Intelligent Wireless Communicatio Systems in 3 Dimensional Environment]

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