New Construction of Quaternary Sequences With Ideal Autocorrelation and Balance Property

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***Abstract*—In this paper, a new construction method of quaternary sequences of even period $2N$ with ideal autocorrelation and balance property is proposed using the inverse Gray mapping and binary sequences of odd period N with ideal autocorrelation. The new quaternary sequences have improved symbol distribution property when compared to existing quaternary sequences with ideal autocorrelation and balance property.**

I. Introduction

Pseudo-random sequences with good autocorrelation property play an important role in designing digital communication systems. The good correlation property guarantees less interferences in the wireless communication systems. Especially, binary and quaternary sequences are preferred because the binary and quadrature modulations are widely used.

In recent years, several methods have been proposed to generate quaternary sequences with ideal autocorrelation from binary sequences with ideal autocorrelation [1], [2]. Although those sequences have good autocorrelation property, they have a weak point in their symbol distribution, that is, the sequences take the symbols 0 and 2 at the even indices and the symbols 1 and 3 at the odd indices. This characteristic reduces the randomness of the sequences.

In this paper, a new construction method of quaternary sequences of even period $2N$ with ideal autocorrelation and balance property is proposed using the inverse Gray mapping and binary sequences of odd period N with ideal autocorrelation. The proposed quaternary sequences have improved symbol distribution property when compared to existing quaternary sequences with ideal autocorrelation and balance property.

II. Preliminaries

Let $g(t)$ be a q-ary sequence of period N for positive integers q and N. Then a sequence $g(t)$ of period N is said to be balanced if the difference among numbers of occurrences of each element in a period is less than or equal to one.

The cross-correlation function of $g_1(t)$ and $g_2(t)$ is defined as

$$R_{g_1g_2}(\tau) = \sum_{t=0}^{N-1} \omega_q^{g_1(t)-g_2(t+\tau)}.$$

where $0 \le \tau < N$ and ω_q is the complex primitive qth root of unity, e.g., $\omega_4 = \sqrt{-1}$. When $g_1(t)$ and $g_2(t)$ are equal, this function is called the autocorrelation function.

It is well known that a binary sequence of odd period N with ideal autocorrelation property has the distribution of autocorrelation values as

$$R_g(\tau) = \begin{cases} N, & \text{once} \\ -1, & N-1 \text{ times.} \end{cases}$$

In addition, the autocorrelation distribution of a quaternary sequence of even period N with ideal autocorrelation and balance property is given as

$$R_g(\tau) = \begin{cases} N, & \text{once} \\ 0, & \frac{N}{2} \text{ times} \\ -2, & \frac{N}{2}-1 \text{ times.} \end{cases}$$

Let $\phi[s_0, s_1]$ be the inverse Gray mapping defined by

$$\phi[s_0, s_1] = \begin{cases} 0, & \text{if } (s_0, s_1) = (0,0) \\ 1, & \text{if } (s_0, s_1) = (0,1) \\ 2, & \text{if } (s_0, s_1) = (1,1) \\ 3, & \text{if } (s_0, s_1) = (1,0). \end{cases}$$

Given two binary sequences $s_0(t)$ and $s_1(t)$ of period N, we can have a quaternary sequence of period N defined by $q(t) = \phi[s_0(t), s_1(t)]$.

Krone and Sarwate derived the relation between the correlations of the binary sequences and those of the quaternary sequences as follows.

Lemma 1: [3] Let $s_0(t)$, $s_1(t)$, $s_2(t)$, and $s_3(t)$ be binary sequences of the same period. Let $q_0(t)$ and $q_1(t)$ be quaternary sequences defined by $q_0(t) = \phi[s_0(t), s_1(t)]$ and $q_1(t) = \phi[s_2(t), s_3(t)]$, respectively. Then cross-correlation function $R_{q_0q_1}(\tau)$ between $q_0(t)$ and $q_1(t)$ is given as

$$R_{q_0q_1}(\tau) = \frac{1}{2}\left\{R_{s_0s_2}(\tau)+R_{s_1s_3}(\tau)+\omega_4\left(R_{s_0s_3}(\tau) - R_{s_1s_2}(\tau)\right)\right\}$$

where $R_{s_is_j}(\tau)$ is the cross-correlation function of $s_i(t)$ and $s_j(t)$.

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In [1], Jang, Kim, Kim, and No proposed a construction method of quaternary sequences. Despite of having the ideal autocorrelation and balance property, this sequence has a weakness in its symbol distribution in the sense that $q(t) \in \{0,2\}$ if $t \equiv 0 \bmod 2$ and $q(t) \in \{1,3\}$ if $t \equiv 1 \bmod 2$. This property reduces the randomness of the sequences.

III. Construction of New Quaternary Sequences

In this section, by using a binary sequence with ideal autocorrelation and inverse Gray mapping, we propose a new construction method of quaternary sequences with ideal autocorrelation and balance property. The autocorrelation distribution of the proposed quaternary sequences is also derived.

Let $b(t)$ be a binary sequence of odd period N with ideal autocorrelation. Let $b_0(t)$ and $b_1(t)$ be the two binary sequences defined as

$$\begin{array}{l} b_0(t) = b(t), \\ b_1(t) = b(-t). \end{array} \tag{1}$$

Then, a new quaternary sequence of period $2N$ is defined as

$$q(t) = \phi[s_0(t), s_1(t)] \tag{2}$$

where $s_0(t)$ and $s_1(t)$ are the binary sequences of period $2N$ defined by

$$\begin{aligned} s_0(t) &= \begin{cases} b_0(t), & \text{for } t \equiv 0 \bmod 2 \\ b_0(t), & \text{for } t \equiv 1 \bmod 2 \end{cases} \\ s_1(t) &= \begin{cases} b_1(t) \oplus 1, & \text{for } t \equiv 0 \bmod 2 \\ b_1(t), & \text{for } t \equiv 1 \bmod 2. \end{cases} \end{aligned}$$

where $\oplus$ denotes modulo 2 addition.

Theorem 2: Let $q(t)$ be the quaternary sequence defined in (2). Then $q(t)$ has the balance property, i.e.,

$$q(t) = \begin{cases} 0, & \frac{N-1}{2} \text{ times} \\ 1, & \frac{N-1}{2} \text{ times} \\ 2, & \frac{N+1}{2} \text{ times} \\ 3, & \frac{N+1}{2} \text{ times.} \end{cases}$$

Proof: Let $B_i, i = 0, 1, 2, 3,$ be the numbers defined by

$$B_i = |\{t | q(t) = i, 0 \le t < 2N\}|.$$

If we define $N_0, N_1, N_2,$ and N_3 as

$$\begin{array}{l} N_0 = |\{t | b_0(t) = 0 \text{ and } b_1(t) = 0, 0 \le t < N\}| \\ N_1 = |\{t | b_0(t) = 0 \text{ and } b_1(t) = 1, 0 \le t < N\}| \\ N_2 = |\{t | b_0(t) = 1 \text{ and } b_1(t) = 1, 0 \le t < N\}| \\ N_3 = |\{t | b_0(t) = 1 \text{ and } b_1(t) = 0, 0 \le t < N\}| \end{array} \tag{3}$$

then, by using the definition in (2), we have

$$B_0 = B_1 = N_0 + N_1 = |\{t | b(t) = 0, 0 \le t < N\}|$$
$$B_2 = B_3 = N_2 + N_3 = |\{t | b(t) = 1, 0 \le t < N\}|.$$

Since all binary sequences with ideal autocorrelation have the balance property, it is clear that

$$\begin{array}{l} N_0 + N_1 = \frac{N-1}{2}, \\ N_2 + N_3 = \frac{N+1}{2}. \end{array}$$

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Theorem 3: Let $q(t)$ be the quaternary sequence defined in (2). Then $q(t)$ has the ideal autocorrelation property with the following distribution

$$R_q(\tau) = \begin{cases} 2N, & \text{for } \tau = 0 \\ 0, & \text{for } \tau \equiv 1 \bmod 2 \\ -2, & \text{for } \tau \equiv 0 \bmod 2 \text{ and } \tau \ne 0. \end{cases}$$

Proof: From Lemma 1, $R_q(\tau)$ can be rewritten as

$$R_q(\tau) = \frac{1}{2}\{R_{s_0}(\tau) + R_{s_1}(\tau) + \omega_4(R_{s_0 s_1}(\tau) - R_{s_1 s_0(\tau)})\}.$$

From the definition of $s_0(t)$ and $s_1(t)$, the autocorrelation function $R_{s_0}(\tau)$ of $s_0(t)$ and the autocorrelation function $R_{s_1}(\tau)$ of $s_1(t)$ are expressed as

$$\begin{aligned} R_{s_0}(\tau) &= 2R_{b_0}(\tau) \\ R_{s_1}(\tau) &= \begin{cases} 2R_{b_1}(\tau) & \text{for } \tau \equiv 0 \bmod 2 \\ -2R_{b_1}(\tau), & \text{for } \tau \equiv 1 \bmod 2 \text{ .} \end{cases} \end{aligned}$$

From the definition of $s_0(t)$ and $s_1(t)$, it is easy to check that

$$s_0(t) + s_1(t+\tau) = s_0(t+N) + s_1(t+N+\tau) \oplus 1, 0 \le t < N,$$

and thus we have

$$R_{s_0 s_1}(\tau) = R_{s_1 s_0}(\tau) = 0.$$

When $b(t)$ has the ideal autocorrelation property, it is clear that

$$R_{b_0}(\tau) = R_{b_1}(\tau) = \begin{cases} N, & \text{for } \tau = 0 \\ -1, & \text{otherwise .} \end{cases}$$

Using this result, $R_{s_0}(\tau)$ and $R_{s_1}(\tau)$ can be calculated as

$$\begin{aligned} R_{s_0}(\tau) &= \begin{cases} 2N, & \text{for } \tau = 0 \text{ or } \tau = N \\ -2, & \text{otherwise} \end{cases} \\ R_{s_1}(\tau) &= \begin{cases} 2N, & \text{for } \tau = 0 \\ -2N, & \text{for } \tau = N \\ -2, & \text{for } \tau \equiv 0 \bmod 2 \text{ and } \tau \ne 0 \\ 2, & \text{for } \tau \equiv 1 \bmod 2 \text{ and } \tau \ne N \text{ .} \end{cases} \end{aligned}$$

Therefore, $R_q(\tau)$ can be derived as

$$\begin{aligned} R_q(\tau) &= \frac{1}{2}\{R_{s_0}(\tau) + R_{s_1}(\tau) + \omega_4(R_{s_0 s_1}(\tau) - R_{s_1 s_0(\tau)})\} \\ &= \begin{cases} 2N, & \text{for } \tau = 0 \\ 0, & \text{for } \tau \equiv 1 \bmod 2 \\ -2, & \text{for } \tau \equiv 0 \bmod 2 \text{ and } \tau \ne 0 \text{ .} \end{cases} \end{aligned}$$

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Note that the quaternary sequence $q(t)$ in (2) is free of the symbol distribution imbalance problem mentioned in the previous section. In fact, the imbalance of symbol distributions between at even t and odd t depends on the base sequence $b(t)$.

References

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