Protograph Design With Multiple Edges for Regular QC LDPC Codes Having Large Girth

Hosung Park, Seokbeom Hong, Jong-Seon No
Department of Electrical Engineering and Computer Science, INMC
Seoul National University
Seoul 151-744, Korea
Email: {lovepark98, fousbyus}@ccl.snu.ac.kr, jsno@snu.ac.kr

Dong-Joon Shin
Department of Electronic Engineering
Hanyang University
Seoul 133-791, Korea
Email: djshin@hanyang.ac.kr

***Abstract*—In this paper, all subgraph patterns of protographs which prevent quasi-cyclic (QC) low-density parity-check (LDPC) codes from having large girth are searched in allowance with multiple edges based on graph theoretic approach. A systematic construction of protographs with multiple edges using combinatorial design is proposed for designing QC LDPC codes with girth greater than or equal to 14.**

I. Introduction

Low-density parity-check (LDPC) codes [1] have been one of major topics for many coding theorists over the past decade due to their near capacity-approaching performance. Since low decoding complexity can be achieved by various iterative decoding algorithms, LDPC codes have been adopted in many practical applications. Especially quasi-cyclic (QC) LDPC codes are well suited for hardware implementation using simple shift registers due to the regularity in their parity-check matrices.

Thorpe [2] introduced the concept of *protograph-based LDPC codes*, a class of LDPC codes lifted from protographs. QC LDPC codes belong to protograph-based LDPC codes because they can be regarded as the lifted ones from the corresponding protographs using cyclic permutations. Therefore, construction of good QC LDPC codes mainly depends on how to design their protographs.

There have been many efforts on the construction of QC LDPC codes with large girth. In [3], some protographs whose lifted codes can have large girth were constructed using combinatorial design by considering only the protographs having a single edge between two nodes. However, the QC LDPC codes lifted from protographs with multiple edges between two nodes can show better performance due to the flexibility in adjusting their degree distribution and have larger minimum distance according to [4]. In this paper, we suggest design methods of protographs with multiple edges for regular QC LDPC codes having large girth by using combinatorial designs.

II. Subgraph Patterns of Protographs

In order for a QC LDPC code to have large girth, its protograph should not have the subgraph patterns which generate short cycles. For this, some subgraph patterns only made up of single edges were searched by brute force method [3]. Since then, systematic approach based on graph theory [5] appeared to classify all subgraph patterns. This method can be applied to the subgraphs with multiple edges as well as single edges.

A. Preliminaries

In this paper, the term ‘protograph’ will represent both the bipartite graph and its incidence matrix based on their equivalence. Let $[p_{ij}]_{ij}$ be a protograph meaning that the horizontal node i and the vertical node j is connected each other via p_{ij} edge(s). If $p_{ij} \geq 2$, there are multiple edges between two nodes in the protograph. As in [5], we define two classes of graphs.

Definition 1 ([5]): A (x_1, x_2, x_3)-*theta graph*, denoted by $T(x_1, x_2, x_3)$, is a graph consisting of two vertices, each of degree three, that are connected to each other via three disjoint paths X_1, X_2, X_3 of the number of edges $x_1 \geq 1$, $x_2 \geq 1$, and $x_3 \geq 1$, respectively. A $(z_1, z_2; y)$-*dumbbell graph*, denoted by $D(z_1, z_2; y)$, is a connected graph consisting of two edge-disjoint cycles Z_1 and Z_2 of the number of edges $z_1 \geq 1$ and $z_2 \geq 1$, respectively, that are connected by a path Y of the number of edges $y \geq 0$.

The girth of a QC LDPC code is determined by the structure of the protograph, lift size, and shift values of each edge. However, we can derive an upper bound on girth of a protograph without lift size determination and shift value assignment by introducing the concept of inevitable cycle [3].

Definition 2: An *inevitable cycle* for a protograph is defined as the shortest one of the cycles which should appear in the QC LDPC code lifted from the protograph regardless of lift size and shift value assignment.

B. Search for Subgraph Patterns

The subgraph pattern P_{2i} is defined as follows: 1) P_{2i} has the inevitable cycle of length $2i$; 2) P_{2i} does not have any subgraph which has an inevitable cycle of length $\geq 2i$; 3) The number of rows is not less than that of columns; 4) For an isomorphic class of graphs, only one matrix P_{2i} must be given as a representative. The next two theorems can be directly derived from results in [5], and their proofs are omitted.

Theorem 1: P_{2i} must be either a theta graph or a dumbbell graph.

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Theorem 2: $T(x_1, x_2, x_3)$ and $D(z_1, z_2; y)$ have the inevitable cycle of length $2(x_1 + x_2 + x_3)$ and $2(z_1 + z_2) + 4y$, respectively.

Now we can find all subgraph patterns including the ones with multiple edges from $T(x_1, x_2, x_3)$ and $D(z_1, z_2; y)$. Since P_{2i}'s are bipartite graphs, the parameters should satisfy the following conditions: 1) $x_1 \geq x_2 \geq x_3 \geq 1$; 2) x_1, x_2, x_3 are all even or odd; 3) $z_1 \geq z_2 \geq 2$, $y \geq 0$; 4) z_1 and z_2 are even. Using the above conditions, all subgraph patterns for inevitable cycles of length up to 16 are listed as belows.

$$P_6 = [3], \qquad P_8 = \begin{bmatrix} 2 & 2 \end{bmatrix}$$

$$P_{10} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}, \; P_{12} = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}, \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

$$P_{14} = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

$$P_{16} = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 1 & 2 \end{bmatrix}, \begin{bmatrix} 2 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}, \begin{bmatrix} 2 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix},$$
$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

Note that the transpose of each subgraph pattern will also cause the inevitable cycle.

III. Construction of Regular Protograph with Inevitable Cycle of Length ≥ 14

Now we will construct regular protographs which avoid some of the listed subgraph patterns. Consider constructing a regular protograph of size $J \times L$ of which the column weight and the row weight are d_v and d_c, respectively. If triple or more multiple edges exist in a protograph, the girth of the lifted QC LDPC code is limited to 6. Therefore protograph construction with only single and double edges will be considered. Let n_2 denote the number of double edges in the protograph. To construct protographs with inevitable cycle of length ≥ 10, the condition that any pair of '2's should not exist in a row and in a column of the protograph is sufficient. And note that the number of double edges are limited to construct a QC LDPC code with large girth.

Lemma 1: If $n_2 > J$, the length of inevitable cycle for the protograph is 8.

In order for a QC LDPC code to have the girth ≥ 12, the protograph should not have P_6, P_8, P_{10}, and their transposes as its subgraphs. The construction of protographs with inevitable cycle of length ≥ 12 is provided in [6] by using balanced ternary design.

Now we will focus on the construction of regular protographs with inevitable cycle of length ≥ 14. A systematic method for the construction of those protographs only with single edges was provided in [3]. However, we should consider the additional two subgraph patterns of P_{12} as well as

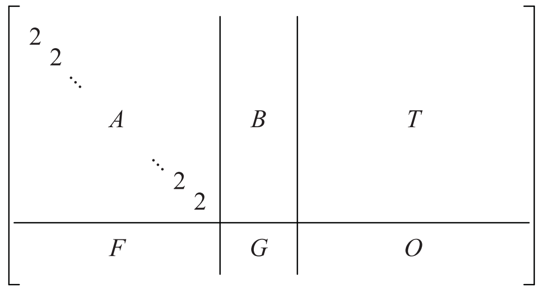

Fig. 1. The structure of regular protograph with inevitable cycle of length greater than or equal to 14.

P_8 and P_{10} having double edges, which make the problem complicated. Assume that $d_v \geq 3$.

Every regular protograph with inevitable cycle of length $\geq$ 14 can be represented by the structure in Fig. 1. The submatrix A of size $n_2 \times n_2$ can have n_2 '2's as its diagonal elements without loss of generality, and the other elements of A should be zero to avoid the first pattern of P_{12}. F is a $(J - n_2) \times n_2$ submatrix and its columns have weight $d_v - 2$. By appropriate column permutation of all but A and F in the protograph, all the columns whose lower part has nonzero weight are located in the section of B and G, and the others consist of T with column weight d_v and zero matrix O. Let $J_G \times L_G$ and $J_T \times L_T$ be the size of G and T, respectively.

A. Regular Protographs with $d_v = 3$

According to Lemma 1, n_2 cannot be greater than J for the inevitable cycle of length $\geq$ 10. Here, we need to find another upper bound on n_2 in regular protographs with $d_v = 3$ for inevitable cycle of length $\geq$ 14. The following theorem provides the maximum number of double edges for those protographs. The proof is omitted and will appear in full version paper.

Theorem 3: Assume that a protograph with $d_v = 3$ and $d_c \geq 4$ has an inevitable cycle of length ≥ 14. Then $n_2 \leq J - 2$.

Necessary conditions on d_c and J for the existence of regular protographs with $d_v = 3$ and $n_2 = J - 2$ to avoid the inevitable cycles of length < 14 can be derived as follows. The proof is omitted and will appear in full version paper.

Theorem 4: Assume that a protograph with $d_v = 3$, $d_c \geq 4$, and $n_2 = J - 2$ has the inevitable cycle of length ≥ 14. Then d_c and J have the following relations:

1) $J \equiv 0 \mod 6 \;\Rightarrow\; d_c = (J-2)/2,\; J/2$
2) $J \equiv 1 \mod 6 \;\Rightarrow\; d_c = (J-1)/2$
3) $J \equiv 2 \mod 6 \;\Rightarrow\; d_c = (J-2)/2$
4) $J \equiv 3 \mod 6 \;\Rightarrow\; d_c = (J-1)/2,\; (J+1)/2$
5) $J \equiv 4 \mod 6 \;\Rightarrow$ None except $J = 10$ and $d_c = 6$
6) $J \equiv 5 \mod 6 \;\Rightarrow\; d_c = (J+1)/2$

Now we will focus on the existence problem and the construction method of the regular protographs with the parameters found in Theorem 4. Note that $J_G = 2$, $L_G =$

$2d_c-(J-2)$, $J_T=J-2$, and $L_T=d_c(J-6)/3$. After determining J and d_c, B, F, and G can be constructed step by step as follows.

1. In F, $J-2$ '1's are placed such that each column has one '1' and two rows have '1's as evenly as possible.
2. In G, the remaining '1's are placed according to the row weight distribution such that each column of G should have one '1'.
3. A pair of '1's in each column of B should be located to avoid P_{10} in the union of A, B, F, and G. At the same time, each row of B should have at most one '1'.

The conditions in Theorem 4 do not guarantee the existence of T. However, we will show that there always exist protographs with all parameters in Theorem 4. Also, construction methods of T will be provided for all $J \geq 9$ using various combinatorial designs.

1) $J \equiv 5 \mod 6$

In this case, we have $d_c=(J+1)/2$, $L_G=3$, and $L_T=(J+1)(J-6)/6$. Given three pairs of '1's in B, we need to construct T of size $(J-2)\times(J+1)(J-6)/6$ to avoid $\begin{bmatrix}1 & 1\\ 1 & 1\end{bmatrix}$ in the union of B and T. For this, Steiner system can be used.

Definition 3 ([7]): A t-(v,k,λ) design is a pair (V,B), where V is a v-set of points and B is a collection of k-subsets (blocks) of V with the property that every t-subset of V is contained in exactly λ blocks in B. A *Steiner system* $S(t,k,v)$ is the t-(v,k,λ) design with $\lambda=1$.

Lemma 2 ([7]): There exists a Steiner system $S(2,3,J-2)$ only when $J \equiv 3,5 \mod 6$.

The number of blocks in $S(2,3,J-2)$ is $(J-2)(J-3)/6$. The incidence matrix of $S(2,3,J-2)$ could be used as T if its two columns are removed. However, it is impossible to avoid repeated pairs in the union of B and T by this method because any two blocks in $S(2,3,J-2)$ cannot have disjoint three pairs. Here we propose a construction method for T in the case of $J \equiv 5 \mod 6$.

1. Remove three columns which form a 6-cycle in the incidence matrix of $S(2,3,J-2)$.
2. Insert one column of weight three where three '1's are located in the rows passed through by the above 6-cycle.
3. Perform row permutation not to have repeated pairs in the union of B and T.

An 11×22 protograph with $d_v=3$ and $n_2=9$ constructed by using $S(2,3,9)$ is shown in Fig. 2. We can check that P_{2i}'s with $i \leq 6$ do not appear in the protograph.

2) $J \equiv 2 \mod 6$

In this case, we have $d_c=(J-2)/2$, $L_G=0$, and $L_T=(J-2)(J-6)/6$. Since B and E do not appear in the protograph, we will only pay attention to avoid repeated pairs just within T where T should be a constant weight (regular) matrix with column weight 3 and row weight $(J-6)/2$. A configuration whose incidence matrix has the column weight

$$\left[\begin{array}{ccccccccc|ccc|cccccccccc}
2&0&0&0&0&0&0&0&0&1&0&0&1&1&1&0&0&0&0&0&0&0\\
0&2&0&0&0&0&0&0&0&1&0&0&0&0&0&0&1&0&1&0&0&1\\
0&0&2&0&0&0&0&0&0&0&1&0&1&0&0&1&1&0&0&0&0&0\\
0&0&0&2&0&0&0&0&0&0&1&0&0&0&0&0&0&1&1&1&0&0\\
0&0&0&0&2&0&0&0&0&0&0&0&0&0&1&1&0&1&0&0&0&1\\
0&0&0&0&0&2&0&0&0&0&0&1&1&0&0&0&0&1&0&0&1&0\\
0&0&0&0&0&0&2&0&0&0&0&1&0&1&0&0&0&0&0&1&0&1\\
0&0&0&0&0&0&0&2&0&0&0&0&0&1&0&1&0&0&1&0&1&0\\
0&0&0&0&0&0&0&0&2&0&0&0&0&0&1&0&1&0&0&1&1&0\\
\hline
1&1&1&1&1&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&1&1&1&1&1&1&0&0&0&0&0&0&0&0&0&0&0
\end{array}\right]$$

Fig. 2. An 11×22 regular protograph with $d_v=3$, $n_2=9$.

3 and the size of $(J-2)\times(J-2)(J-6)/6$ can be used as T.

Definition 4 ([7]): A *configuration* (v_r,b_k) is an incidence structure of v points and b blocks such that 1) each block contains k points, 2) each point lies on r blocks, 3) two different points are connected by at most one block. If $v=b$ and hence $r=k$, then configuration is *symmetric* and denoted by v_k.

It is important to check the existence of the configuration with the required parameters. The following theorem shows that T can always be constructed from the configuration.

Theorem 5: There exists a configuration (v_r,b_k) with $v=J-2, b=(J-2)(J-6)/6, k=3, r=(J-6)/2$ for all $J \equiv 2 \mod 6$ and $J \geq 14$.

Proof: Necessary conditions for the existence of (v_r,b_k) configuration [8] are as the followings: 1) $v \leq b$ and $k \leq r$, 2) $vr=bk$, 3) $v \geq r(k-1)+1$. We can easily check that the parameters in the theorem satisfy these conditions. Finally, the existence of the configuration is guaranteed according to Theorem 3.1 in [8], that is, there exists a configuration with $k=3$ iff the necessary condition holds. ■

The configuration (v_r,b_k) with $v=J-2, b=(J-2)(J-6)/6, k=3, r=(J-6)/2$ can be constructed by using the method in [8].

3) $J \equiv 3 \mod 6$

(i) $d_c=(J-1)/2$

In this case, we have $L_G=1$ and $L_T=(J-1)(J-6)/6$. Then B has only one pair of '1's. Since a Steiner system $S(2,3,J-2)$ with $(J-2)(J-3)/6$ blocks exists, T can be constructed by removing $J/3$ columns from the incidence matrix of $S(2,3,J-2)$. However, we should check whether there always exist $J/3$ columns in the incidence matrix satisfying the following requirements: In the matrix consisting of these $J/3$ columns, 1) two rows have weight 2 and the others have weight 1, and 2) one of the $J/3$ columns has three '1's at the two rows of weight 2 and another row.

Definition 5 ([7]): A *parallel class* in a design is a set of blocks that partition the point set. A *resolvable design* is a design whose blocks can be partitioned into parallel classes.

It is known that for $J \equiv 3 \mod 6$, $S(2,3,J-2)$ contains as substructure a configuration (v_r,b_k) with $v=J-3, b=(J-3)(J-5)/6, k=3, r=(J-5)/2$ which has at least

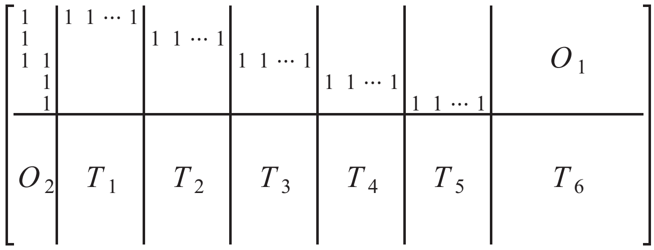

Fig. 3. The structure of the incidence matrix of $OP(J-2)$ for $J \equiv 1 \mod 6$.

$$
\left[\begin{array}{ccccccccccccc|c|ccccccccccccccccccccc}
2&0&0&0&0&0&0&0&0&0&0&0&0&1&1&1&1&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&2&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&1&0&0&0&0&0&0&0&0&1&1&0&0&1&0&0\\
0&0&2&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&1&1&1&1&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&2&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&1&1&1&1&0&0&0&0&0&0&0&0&0\\
0&0&0&0&2&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&1&0&0&0&1&1&1&0&0&0&0&0&0\\
0&0&0&0&0&2&0&0&0&0&0&0&0&0&0&1&0&0&1&0&0&0&0&1&0&0&0&0&0&1&1&0&0&0&0\\
0&0&0&0&0&0&2&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&1&0&0&0&0&0&0&0&1&1&1&0\\
0&0&0&0&0&0&0&2&0&0&0&0&0&0&0&0&1&0&0&1&0&0&0&0&1&0&1&0&0&0&0&0&0&0&1\\
0&0&0&0&0&0&0&0&2&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&1&0&1&0&0&0&1&0&0&1\\
0&0&0&0&0&0&0&0&0&2&0&0&0&0&0&0&0&0&0&0&1&0&1&0&0&0&0&0&0&1&0&0&0&1&1\\
0&0&0&0&0&0&0&0&0&0&2&0&0&0&0&0&0&0&0&0&0&1&0&0&1&0&0&1&0&0&1&0&1&0&0\\
0&0&0&0&0&0&0&0&0&0&0&2&0&0&0&0&0&1&0&0&1&0&0&0&0&0&1&0&0&0&1&1&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&2&0&0&0&0&1&0&0&0&1&0&0&0&1&0&0&1&0&0&0&0&1&0\\
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1&1&1&1&1&1&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&1&1&1&1&1&1&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0
\end{array}\right]
$$

Fig. 4. A 13×26 regular protograph with $d_v = 3$, $n_2 = 11$.

one parallel class. This property implies that there are $J/3-1$ blocks in $S(2,3,J-2)$ which partition all but one point. And there is another block in $S(2,3,J-2)$, which includes that point. Consequently, we propose the following construction method for T in the case of $J \equiv 3 \mod 6$ and $d_c = (J-1)/2$.

1. Construct $S(2,3,J-2)$.
2. Select one row in the incidence matrix of $S(2,3,J-2)$ such that if the row and its incident columns are removed from the matrix, the remnant forms the incidence matrix of a configuration (v_r, b_k) with $v = J-3, b = (J-3)(J-5)/6, k = 3, r = (J-5)/2$ including at least one parallel class.
3. Find $J/3-1$ columns which form one parallel class in the configuration.
4. In the incidence matrix of $S(2,3,J-2)$, remove a column incident to the selected row and $J/3-1$ columns corresponding to the parallel class of the configuration.
5. Perform row permutation not to have repeated pairs in the union of B and T.

Note that the above construction method cannot be applied to the case of $J = 9$ and $d_c = 4$ because the configuration $(6_2, 4_3)$ does not have a parallel class.

(ii) $d_c = (J+1)/2$

In this case, we have $L_G = 3$ and $L_T = (J+1)(J-6)/6$. There are three pairs of '1's in B. Since $S(2,3,J-2)$ exists for $J \equiv 3 \mod 6$, the construction method in the case of $J \equiv 5 \mod 6$ can be applied in the same way.

4) $J \equiv 1 \mod 6$

Definition 6 ([7]): Let K be a subset of positive integers and let λ be a positive integer. A *pairwise balanced design* of order v with block sizes from K, denoted by PBD$(v, K; \lambda)$, is a pair $(\mathcal{V}, \mathcal{B})$, where $\mathcal{V}$ is a point set of cardinality v and $\mathcal{B}$ is a family of blocks of $\mathcal{V}$ which satisfy the properties: 1) If $B \in \mathcal{B}$, then $|B| \in K$, 2) Every pair of distinct elements of $\mathcal{V}$ occurs in exactly λ blocks of $\mathcal{B}$.

In this case, we have $d_c = (J-1)/2$, $L_G = 1$, and $L_T = (J-1)(J-6)/6$. Let $PBD(v, K)$ denote a $PBD(v, K; \lambda)$ with $\lambda = 1$ and use $PBD(v, K \cup k^\star)$ to denote a PBD containing only one block of size k in the PBD where $k \notin K$ is a positive integer. Let $OP(v)$ denote the 2-$(v,3,1)$ optimal packing. According to the construction of optimal packings [7], a matrix of size $(J-2) \times (J^2-5J-2)/6$ with column weight 3 can be obtained from $PBD(J-2, \{3, 5^\star\})$ by replacing the block of size 5 with two blocks of size 3. The incidence matrix of this optimal packing can be represented by the structure in Fig. 3. O_1 and O_2 are zero matrices. T_i $(i = 1, \cdots, 5)$ is the regular matrix of size $(J-7) \times (J-7)/2$ with column weight 2, and T_6 is the regular matrix of size $(J-7) \times (J-7)(J-13)/6$ with column weight 3. The union of T_i's $(i = 1, \cdots, 6)$ forms another $PBD(J-7, \{2,3\})$ and each T_i $(i = 1, \cdots, 5)$ corresponds to one parallel class.

Conjecture 1: There exists a $PBD(v, \{2,3\})$ for all $v \equiv 0 \mod 6$, $v \geq 6$ such that 1) all pairs exactly form five parallel classes, and 2) there is a parallel class consisting of three pairs each of which belongs to distinct one of the five parallel classes and $(v-6)/3$ triples.

Based on this conjecture, we propose a construction method of T for $J \equiv 1 \mod 6$. The incidence matrix of $OP(J-2)$ will be modified in order to have the row weights and the size adequate for T and to avoid the repeated pair in the union of B and T at the same time.

1. Construct $PBD(J-7, \{2,3\})$ satisfying the conditions in Conjecture 1.
2. To construct $PBD(J-2, \{3, 5^\star\})$, permute all columns of the incidence matrix of $PBD(J-7, \{2,3\})$ to satisfy the following conditions: 1) Each parallel class with columns of weight 2 forms one of T_i's $(i = 1, \cdots, 5)$, 2) all columns of weight 3 form T_6, and 3) among three columns of weight 2 in the parallel class of Conjecture 1, one column is located in either T_1 or T_2, another in T_3, and the other in either T_4 or T_5.
3. Obtain $OP(J-2)$ from $PBD(J-2, \{3, 5^\star\})$ by replacing the block of size 5 with two blocks of size 3.
4. In the incidence matrix of $OP(J-2)$, remove $(J-4)/3$ columns corresponding to the parallel class of $PBD(J-7, \{2,3\})$.
5. Perform row permutation not to have repeated pairs in the union of B and T.

A protograph of size 13×26 with $d_v = 3$ and $n_2 = 11$ is shown in Fig. 4.

5) $J \equiv 0 \mod 6$

(i) $d_c = (J-2)/2$

In this case, we have $L_G = 0$ and $L_T = (J-2)(J-6)/6$. As the case of $J \equiv 2 \mod 6$, the incidence matrix of a configuration (v_r, b_k) with $v = J-2, r = (J-6)/2, b = (J-2)(J-6)/6, k = 3$ can be used as T. The existence of the configuration is guaranteed by Theorem 5 and this configuration can be constructed by using difference triangle set (see [8]).

(ii) $d_c = J/2$

In this case, we have $L_G = 2$ and $L_T = J(J-6)/6$ and we can construct T from $OP(J-2)$.

When $J \equiv 0 \mod 6$, $OP(J-2)$ can be obtained from $OP(J-1)$ by removing the first row and its incident columns in the incidence matrix of $OP(J-1)$. We can see that the size of the incidence matrix of $OP(J-2)$ is $(J-2) \times (J^2 - 6J+6)/6$. Hence T can be constructed by removing proper one column from the incidence matrix. Here we propose a construction method of T for $J \equiv 0 \mod 6$ and $d_c = J/2$.

1. Construct $OP(J-1)$ in the same way as the case of $J \equiv 1 \mod 6$.
2. Obtain $OP(J-2)$ from $OP(J-1)$ by removing the first row and its incident columns in the incidence matrix of $OP(J-1)$.
3. Remove the first column in the incidence matrix of $OP(J-2)$.
4. Perform row permutation not to have repeated pairs in the union of B and T.

6) $J = 9, 10$

When $J = 9$ and $d_c = 4$, the construction method explained in the case of $J \equiv 3 \mod 6$ and $d_c = (J-1)/2$ cannot be applied as mentioned before. However, we can easily find T by hand. When $J = 9$ and $d_c = 5$, T can be constructed by using $S(2,3,7)$ in the same way as the case of $J \equiv 3 \mod 6$ and $d_c = (J+1)/2$. When $J = 10$ and $d_c = 6$, the incidence matrix of a configuration $(8_3, 8_3)$ can be used as T.

Until now, we proposed the construction methods for all regular protographs with $d_v = 3$ and $n_2 = J-2$. Protographs with $d_v = 3$ and $n_2 < J-2$ can be constructed in a similar way. In F, the distribution of '1's is more flexible as long as each column of F has one '1'. G may have columns of weight 1, 2, or 3 and should not contain the third pattern of P_{12}. B is determined in the way that the union of A, F, G, and B should not contain P_{10}, and the second and third patterns of P_{12}. Then the construction of T is the problem of 2-$(n_2, 3, 1)$ packing with the forbidden pairs in B.

Definition 7 ([7]): Let $v \geq k \geq t$. A t-(v, k, λ) *packing* is a pair $(X, \mathcal{B})$, where X is a v-set of points and $\mathcal{B}$ is a collection of k-subsets of X (blocks), such that every t-subset of points occurs in at most λ blocks in $\mathcal{B}$. The *packing number* $D_\lambda(v, k, t)$ is the maximum number of blocks in any t-(v, k, λ) packing. A t-(v, k, λ) packing $(X, \mathcal{B})$ is *optimal* if $|\mathcal{B}| = D_\lambda(v, k, t)$.

B. Regular Protographs with $d_v \geq 4$

In the case of $d_v \geq 4$, the protograph with inevitable cycle of length ≥ 14 also has the structure in Fig. 1. Each column of F has $d_v - 2$ '1's, and there should be no repeated pair in F to avoid the second pattern of P_{12}. G may have columns of weight i $(1 \leq i \leq d_v)$ and should not have the second and third patterns of P_{12} in the union of A, F, and G. Determination of B and T is the same as the case of $d_v = 3$ and $n_2 < J-2$ except that 2-$(n_2, d_v, 1)$ packing is used instead. Note that T can be constructed systematically by using combinatorial designs if it has the similar size to the optimal packing.

IV. Conclusion

The subgraph patterns of protographs which cause inevitable cycles were fully searched by using the graph theoretic approach in allowance with multiple edges in the protographs. To have regular QC LDPC codes with girth greater than or equal to 14, we provided the construction methods for all protographs with column weight three and the maximum number of double edges case by case. These construction methods can be extended to the protographs with other parameters.

Acknowledgment

This work was supported by the National Research Foundation of Korea (NRF) grant funded by the Korea government (MEST) (No.2011-0000328) and also supported by the KCC (Korea Communications Commission) under the R&D program supervised by the KCA (Korea Communications Agency) (KCA-2011-08913-04003).

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