A New Parity Structure With Multi-Weight Circulants for QC LDPC Codes

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***Abstract*—The block dual-diagonal (BDD) parity structure is widely adopted in many practical irregular quasi-cyclic (QC) low-density parity-check (LDPC) codes. These QC LDPC codes have good error-correcting performance in waterfall region but usually show relatively high error floors in low error rate region. In this paper, by using multi-weight circulants, a new BDD structure is proposed for the parity part of irregular QC LDPC codes to lower error floors and support efficient encoding. Since the parity part of parity-check matrices has flexible degree distribution with the aid of multi-weight circulants, QC LDPC codes with the proposed BDD structure can have large minimum Hamming distance compared to those with the conventional BDD structure, especially, in the low-rate case. Simulation results show that QC LDPC codes with the proposed BDD structure have lower error floor than those with the conventional BDD structure.**

I. Introduction

Irregular low-density parity-check (LDPC) codes generally outperform regular LDPC codes in waterfall region and have a good threshold [1]. However, those LDPC codes of finite length are prone to suffer from high error floors [2] in low error rate region. The error floor phenomenon is closely related to minimum Hamming distance and some combinatorial structures such as stopping sets [3] and trapping sets [2]. In designing LDPC codes for low error floors, small minimum Hamming distance and small-size stopping sets or trapping sets should be avoided. Most well-known irregular LDPC codes have many degree-2 variable nodes [1], [4], [5] but it is known that the minimum Hamming distance of a finite-length irregular LDPC code tends to get smaller as it has more degree-2 variable nodes [6], [7].

Recently, irregular quasi-cyclic (QC) LDPC codes with the block dual-diagonal (BDD) parity structure are widely adopted in many standards such as IEEE 802.16e [4] and 802.11n [5] because the BDD structure enables the efficient encoding [8], [9] and leads to good error-correcting performance especially in waterfall region. The BDD structure consists of mostly degree-2 variable nodes and therefore, if the rate of QC LDPC codes with the BDD structure is given, the fraction of degree-2 variable nodes in the parity-check matrices cannot be below a certain value. This restriction may prevent QC LDPC codes from having good degree distributions. Especially, low-rate QC LDPC codes with the BDD structure have so many degree-2 variable nodes that they cannot have large minimum Hamming distance for finite-length case and good threshold for asymptotic case [1].

In this paper, a new BDD structure is proposed for irregular QC LDPC codes to lower error floors. By adopting multi-weight circulants, the parity part of parity-check matrices can have flexible degree distribution while maintaining BDD structure and supporting the efficient encoding. Therefore, by properly adjusting the fraction of degree-2 variable nodes, QC LDPC codes can have large minimum Hamming distance and good threshold. The proposed BDD structure improves the error-correcting performance, especially, in the case of low-rate QC LDPC codes. QC LDPC codes with the proposed BDD structure which do not include harmful stopping sets or trapping sets are generated and it is shown via numerical analysis that they have lower error floor than QC LDPC codes with the conventional BDD structure.

II. A New Block Dual-Diagonal Parity Structure

A. Quasi-Cyclic LDPC Codes

Let $\mathcal{C}$ be a binary QC LDPC code whose parity-check matrix $\mathbf{H}$ is a $J \times L$ array of $z \times z$ circulants or zero matrices as

$$\mathbf{H} = \begin{bmatrix} \mathbf{H}_{0,0} & \mathbf{H}_{0,1} & \cdots & \mathbf{H}_{0,L-1} \\ \mathbf{H}_{1,0} & \mathbf{H}_{1,1} & \cdots & \mathbf{H}_{1,L-1} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{H}_{J-1,0} & \mathbf{H}_{J-1,1} & \cdots & \mathbf{H}_{J-1,L-1} \end{bmatrix}$$

where a *circulant* $\mathbf{H}_{j,l}$ is defined as a matrix whose each row is a cyclic shift of the row above it. Let $M \times N$ be the size of $\mathbf{H}$. Then $M = Jz$, $N = Lz$, and the code design rate $R = 1 - M/N = 1 - J/L$. The *Tanner graph* of $\mathcal{C}$ is a bipartite graph which has $\mathbf{H}$ as its incidence matrix.

The *weight* of a circulant $\mathbf{H}_{j,l}$ is defined as the number of nonzero elements in the first column and denoted by $\mathrm{wt}(\mathbf{H}_{j,l})$. A circulant of weight 1 is called *circulant permutation matrix*. A *multi-weight circulant* is defined as a circulant of weight larger than 1. A circulant is entirely described by the positions of nonzero elements in the first column. Let the index of the i-th element in the first column be $i-1$ for $1 \leq i \leq z$. The *shift value*(s) of a circulant is defined as the index (indices) of the nonzero element(s) in the first column. Note that a shift value

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$$\mathbf{P}_p = \begin{bmatrix} 2 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 2 & 1 \\ 1 & 0 & 0 & 0 & \cdots & 0 & 0 & 1 \end{bmatrix}, \quad \mathbf{H}_p(x) = \begin{bmatrix} x^0 + x^{s_1} & x^0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & x^0 + x^{s_2} & x^0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & x^0 & x^0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & x^0 + x^{s_3} & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & x^0 & x^0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & x^0 + x^{s_{n_2}} & x^0 \\ x^0 & 0 & 0 & 0 & \cdots & 0 & 0 & x^0 \end{bmatrix}$$

takes the value from 0 to $z-1$ and a multi-weight circulant has multiple shift values.

QC LDPC codes can be fully represented by binary polynomials as shown in [10]. This polynomial representation is based on the isomorphism between $z \times z$ binary circulants and the polynomial ring $\mathbb{F}_2[x]/(x^z+1)$. The *polynomial parity-check matrix* $\mathbf{H}(x)$ of $\mathcal{C}$ is defined as

$$\mathbf{H}(x) = \begin{bmatrix} h_{0,0}(x) & h_{0,1}(x) & \cdots & h_{0,L-1}(x) \\ h_{1,0}(x) & h_{1,1}(x) & \cdots & h_{1,L-1}(x) \\ \vdots & \vdots & \ddots & \vdots \\ h_{J-1,0}(x) & h_{J-1,1}(x) & \cdots & h_{J-1,L-1}(x) \end{bmatrix}$$

where $h_{j,l}(x) = \sum_{i=0}^{z-1} h_{j,l,i} x^i \in \mathbb{F}_2[x]/(x^z+1)$ and $h_{j,l,i}$ is the element with the index i in the first column of $\mathbf{H}_{j,l}$. We can see that the number of terms in $h_{j,l}(x)$ is equal to $\mathrm{wt}(\mathbf{H}_{j,l})$ and the degrees of all terms in $h_{j,l}(x)$ are equivalent to the shift values of $\mathbf{H}_{j,l}$.

QC LDPC codes are also described as *protograph-based LDPC codes* [11] because they can be regarded as the lifted ones from the protographs using cyclic permutations. The incidence matrix of the protograph $\mathbf{P}$ of $\mathcal{C}$ is represented as $\mathbf{P} = [p_{j,l}]$, where $p_{j,l} = \mathrm{wt}(\mathbf{H}_{j,l})$. By abuse of notation, $\mathbf{P}$ will represent the bipartite graph itself or its incidence matrix based on their equivalence. Let z be called the *lift size* of $\mathcal{C}$. A multi-weight circulant $\mathbf{H}_{j,l}$ is represented as parallel edges between the horizontal node j and the vertical node l in the protograph and each edge in the protograph has a shift value so that the edge is lifted by using the cyclic permutation with the shift value to generate $\mathcal{C}$.

For convenience, each matrix is divided into the message part and the parity part: $\mathbf{H} = [\ \mathbf{H}_m \mid \mathbf{H}_p\ ]$, $\mathbf{H}(x) = [\ \mathbf{H}_m(x) \mid \mathbf{H}_p(x)\ ]$, and $\mathbf{P} = [\ \mathbf{P}_m \mid \mathbf{P}_p\ ]$. Note that the size of $\mathbf{H}_m$ and $\mathbf{H}_p$ is $M \times (N-M)$ and $M \times M$, respectively, and the size of $\mathbf{H}_m(x)$ and $\mathbf{P}_m$ is $J \times (L-J)$ and the size of $\mathbf{H}_p(x)$ and $\mathbf{P}_p$ is $J \times J$. Accordingly, a codeword of $\mathcal{C}$ is split into the message part $\mathbf{m}$ and the parity part $\mathbf{p}$.

B. The Proposed Block Dual-Diagonal Structure

We propose a new BDD structure for the parity part of the parity-check matrix, which has the forms in the protograph and the polynomial parity-check matrix as in the top of this page.

In the proposed $\mathbf{P}_p$, 2's are located only at diagonal positions. Let n_2, $1 \le n_2 \le J$, be the number of 2's in $\mathbf{P}_p$. Clearly, n_2 equals to the number of weight-2 circulants in $\mathbf{H}_p$ or the number of binomials in $\mathbf{H}_p(x)$. Since n_2 is not a fixed number, it is usually determined by the desired degree distribution. In each weight-2 circulant, one has zero shift value and the other has a nonzero shift value from $\{1, 2, \ldots, z-1\}$. Let r_i and s_i, $i = 1, 2, \ldots, n_2$, denote the row index (equal to the column index) and the nonzero shift value of the i-th binomial in $\mathbf{H}_p(x)$, respectively. Note that $0 \le r_i \le J-1$, $1 \le s_i \le z-1$, and $r_1 < r_2 < \cdots < r_{n_2}$. Define S as the multiset of all nonzero shift values in $\mathbf{H}_p$, that is, $S = \{s_i | i = 1, \ldots, n_2\}$. Note that an element is allowed to appear more than once in a multiset.

Unlike the conventional BDD structure, the proposed BDD structure provides the flexibility in controlling the fraction of degree-2 and degree-3 variable nodes by adjusting n_2. Note that for given J and z, the fraction of degree-2 variable nodes for the rate-R QC LDPC codes with the proposed BDD structure is $(1-R)(1-n_2/J)$ if $\mathbf{H}_m$ does not have degree-2 variable nodes. Even though R is small, the fraction of degree-2 variable nodes can be decreased by increasing n_2. Thus QC LDPC codes with the proposed BDD structure have more flexibility in taking any good degree distribution, which is a desirable property for the irregular QC LDPC codes because good irregular QC LDPC codes generally have many degree-3 variable nodes as well as degree-2 variable nodes as seen in optimal degree distributions [1] and many standards [4], [5], [12]. Also, QC LDPC codes with the proposed BDD structure can have larger minimum Hamming distance and better threshold than those with the conventional BDD structure.

III. Efficient Encoding of QC LDPC Codes With the Proposed BDD Structure

The proposed BDD structure can provide the efficient encoding by properly determining the nonzero shift values of weight-2 circulants. Assume that n_2 is already determined according to the desired degree distribution, and J and z are also given. S is firstly determined for the efficient encoding in this section and then r_i, $i = 1, \ldots, n_2$ and the ordering of the elements in S are also determined in Section IV.

Recall the efficient encoding of LDPC codes in [8]. A parity-check matrix $\mathbf{H}$ can be partitioned into

$$\mathbf{H} = \begin{bmatrix} \mathbf{A} & \mathbf{B} & \mathbf{T} \\ \mathbf{C} & \mathbf{D} & \mathbf{E} \end{bmatrix}$$

$$\mathbf{T}^{-1}(x) = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ 1+x^{s_2} & 1 & 0 & \cdots & 0 & 0 \\ (1+x^{s_2})(1+x^{s_3}) & 1+x^{s_3} & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ (1+x^{s_2})\cdots(1+x^{s_{J-2}}) & (1+x^{s_3})\cdots(1+x^{s_{J-2}}) & (1+x^{s_4})\cdots(1+x^{s_{J-2}}) & \cdots & 1 & 0 \\ (1+x^{s_2})\cdots(1+x^{s_{J-1}}) & (1+x^{s_3})\cdots(1+x^{s_{J-1}}) & (1+x^{s_4})\cdots(1+x^{s_{J-1}}) & \cdots & 1+x^{s_{J-1}} & 1 \end{bmatrix}$$

where $\mathbf{B}$, $\mathbf{T}$, $\mathbf{D}$, and $\mathbf{E}$ form the parity part, $\mathbf{T}$ is a lower-triangular matrix, and $\mathbf{D}$ is a square matrix. Denote the parity part of a codeword by $\mathbf{p} = [\ \mathbf{p}_1\ \mathbf{p}_2\ ]$, where $\mathbf{p}_1$ corresponds to $\mathbf{B}$ and $\mathbf{D}$ and $\mathbf{p}_2$ corresponds to $\mathbf{T}$ and $\mathbf{E}$. Then, $\mathbf{p}_1$ and $\mathbf{p}_2$ are obtained as

$$\mathbf{p}_1^T = \boldsymbol{\phi}^{-1}(\mathbf{E}\mathbf{T}^{-1}\mathbf{A} + \mathbf{C})\mathbf{m}^T$$
$$\mathbf{p}_2^T = \mathbf{T}^{-1}(\mathbf{A}\mathbf{m}^T + \mathbf{B}\mathbf{p}_1^T)$$

where $\boldsymbol{\phi} := \mathbf{E}\mathbf{T}^{-1}\mathbf{B} + \mathbf{D}$.

Now, consider the efficient encoding of QC LDPC codes with the proposed BDD structure. First, assume that $n_2 = J$, which means every diagonal element in $\mathbf{P}_p$ is 2. The case of other value of n_2 can be treated similarly as this case. Let $\mathbf{H}(x)$ be represented by a partitioned polynomial parity-check matrix as

$$\mathbf{H}(x) = \begin{bmatrix} \mathbf{A}(x) & \mathbf{B}(x) & \mathbf{T}(x) \\ \mathbf{C}(x) & \mathbf{D}(x) & \mathbf{E}(x) \end{bmatrix}.$$

From the definition of the proposed BDD structure, we have $\mathbf{B}(x) = \begin{bmatrix} 1+x^{s_1} & 0 & \cdots & 0 \end{bmatrix}^T$, $\mathbf{D}(x) = \begin{bmatrix} 1 \end{bmatrix}$, $\mathbf{E}(x) = \begin{bmatrix} 0 & \cdots & 0 & 1+x^{s_J} \end{bmatrix}$, and

$$\mathbf{T}(x) = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ 1+x^{s_2} & 1 & 0 & \cdots & 0 & 0 \\ 0 & 1+x^{s_3} & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & 0 & \cdots & 1+x^{s_{J-1}} & 1 \end{bmatrix}.$$

We can easily derive the inverse of $\mathbf{T}(x)$ as in the top of this page. Finally, $\boldsymbol{\phi}(x)$ is derived as

$$\begin{aligned} \boldsymbol{\phi}(x) &= \mathbf{E}(x)\mathbf{T}^{-1}(x)\mathbf{B}(x) + \mathbf{D}(x) \\ &= (1+x^{s_1})(1+x^{s_2})\cdots(1+x^{s_J}) + 1. \end{aligned}$$

Therefore, $\boldsymbol{\phi}$ is a $z \times z$ circulant of weight equal to the number of terms in $\boldsymbol{\phi}(x)$.

For the efficient encoding, the existence of $\boldsymbol{\phi}^{-1}$ should be guaranteed. A simple way for this is making $\boldsymbol{\phi}$ be a circulant permutation matrix, that is, $\boldsymbol{\phi}(x) = x^k$ for some $0 \le k \le z-1$, by finding proper s_i, $i = 1, 2, \ldots, J$. Then we have $\boldsymbol{\phi}^{-1}(x) = x^{-k \mod z}$ and $\mathbf{p}_1$ and $\mathbf{p}_2$ can be efficiently encoded. We can find such a S from Theorems 1 and 2 whose proofs are omitted in this paper. Note that when z is even, s_i should not be $z/2$ to avoid cycles of length 4 inside the i-th weight-2 circulant of $\mathbf{H}_p$ [10].

Theorem 1: Assume that $z \neq 2^k$ for all integers $k < J$. Then $S = \{2^0 \mod z, 2^0 \mod z, 2^1 \mod z, 2^2 \mod z, \ldots, 2^{J-2} \mod z\}$ yields $\boldsymbol{\phi}(x) = x^{2^{J-1} \mod z}$ and guarantees that cycles of length 4 do not occur in each weight-2 circulant.

Theorem 2: Assume that $z = 2^l$ for some integer $\log_2 J + 1 \le l \le J-1$. Then, there exist s_i, $i = 1, \ldots, J$, which generate $\boldsymbol{\phi}(x) = x^{2^{l-1}}$ and avoid cycles of length 4 in each weight-2 circulant.

From Theorems 1 and 2, we can clearly determine S supporting the efficient encoding and avoid cycles of length 4 in each weight-2 circulant for $n_2 = J$ and almost all z. Note that Theorems 1 and 2 hold regardless of r_i, $i = 1, \ldots, J$ and in addition, it is not necessary to know every s_i. For the efficient encoding, it is only required to determine the multiset S from which each s_i will be taken in Section IV.

In the case of $n_2 < J$, we can similarly have $\boldsymbol{\phi}(x) = (1+x^{s_1})(1+x^{s_2})\cdots(1+x^{s_{n_2}}) + 1$ regardless of r_i, $i = 1, \ldots, n_2$. Hence, $\{2^0 \mod z, 2^0 \mod z, 2^1 \mod z, 2^2 \mod z, \ldots, 2^{n_2-2} \mod z\}$ or the splitted version of $\{2^0, 2^0, 2^1, 2^2, \ldots, 2^{l-2}\}$ for an integer $l < n_2$ as in Theorems 1 and 2 can be used as S to support the efficient encoding and avoid cycles of length 4.

IV. Search for Good QC LDPC Codes With the Proposed BDD Structure

It is experimentally shown in [9] that the girth of the subgraph consisting of only variable nodes of low degree (i.e., degrees 2 and 3) in $\mathbf{H}$ is desired to be large for good error-correcting performance in the error floor region. This observation agrees with the concept of the approximate cycle extrinsic message degree (ACE) algorithm [13] that are used to avoid harmful stopping sets and trapping sets. Accordingly, for the given S in Theorem 1 or 2, we aim to find r_i, $i = 1, \ldots, n_2$ and the ordering of the elements in S for sufficiently large girth of $\mathbf{H}_p$.

The girth of a QC LDPC code is clearly determined by the structure of the protograph, lift size, and all shift values assigned to edges. However, we can derive an upper bound on the girth of QC LDPC codes lifted from a given protograph without considering the lift size and shift values based on the concept of inevitable cycles.

Definition 1 ([14], [15]): An *inevitable cycle* induced by a protograph is defined as the shortest one of the cycles which should appear in the QC LDPC codes lifted from the protograph regardless of the lift size and shift values.

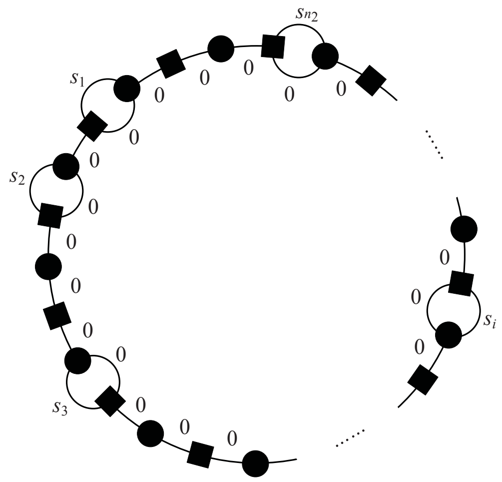

Fig. 1. Graphical representation of the explanatory protograph $\mathbf{P}_p$ in Subsection II-B. Filled circles and squares represent vertical and horizontal nodes, respectively.

The protograph of the proposed BDD structure is relatively simple as shown in Fig. 1.

Definition 2 ([16]): A $(x_1, x_2; y)$-*dumbbell graph*, denoted by $D(x_1, x_2; y)$, is a connected graph consisting of two edge-disjoint cycles X_1 and X_2 of the numbers of edges $x_1 \geq 1$ and $x_2 \geq 1$, respectively, that are connected by a path Y of the number of edges $y \geq 0$.

Lemma 1 ([16]): $D(x_1, x_2; y)$ induces the inevitable cycles of length $2(x_1 + x_2) + 4y$.

In the protograph of the proposed BDD structure, we can find n_2 dumbbell graphs connected back to back to form a chain. Let y_i, $i = 1, \ldots, n_2 - 1$, denote the number of 1's along with the shortest path between the i-th and $(i+1)$-th 2's in $\mathbf{P}_p$ and y_{n_2} denote the one between n_2-th and the first 2's. Then, we have $y_i = 2(r_{i+1} - r_i) - 1$ for $i = 1, \ldots, n_2 - 1$ and $y_{n_2} = 2(J + r_1 - r_{n_2}) - 1$. The length of the inevitable cycles induced by $\mathbf{P}_p$ can be obtained in the next theorem.

Theorem 3: The length of the inevitable cycles induced by $\mathbf{P}_p$ is $\min_{i:1 \leq i \leq n_2} \{8 + 4y_i\}$.

Proof: In $\mathbf{P}_p$, there are n_2 dumbbell graphs, that is, $D(2, 2; y_i)$ for $i = 1, \ldots, n_2$. Based on the Definition 1, the inevitable cycles induced by $\mathbf{P}_p$ are determined only by $D(2, 2; y_i)$ with the smallest y_i. From Lemma 1, the length of the inevitable cycles is $\min_{i:1 \leq i \leq n_2} \{8 + 4y_i\}$. ∎

According to Theorem 3, y_i's, $i = 1, \ldots, n_2$, are desired to have as close values as possible for avoiding short inevitable cycles in $\mathbf{H}_p$. Although the length of the inevitable cycles induced by $\mathbf{P}_p$ is an upper bound on the girth of $\mathbf{H}_p$, not the girth itself, these y_i's are adequate for achieving sufficiently large girth. In addition, since a cycle in $\mathbf{H}_p$ is generated by a walk traversing two or more double edges in $\mathbf{P}_p$ [15], a small y_i is more likely to generate short cycles. For y_i's determined as equally as possible, the ordering of the elements in S which maximizes the girth of $\mathbf{H}_p$ can be obtained via extensive search. This full search is a time-consuming job but for moderate n_2, it is generally manageable because n_2 is not large in most cases.

TABLE I
DEGREE DISTRIBUTIONS OF RATE-$1/3$ QC LDPC CODES BASED ON 30×45 PROTOGRAPHS WITH THE CONVENTIONAL AND THE PROPOSED BDD STRUCTURES

Conv	d_v	2	3	12	d_c	5	6
	# of cols	29	11	5	# of rows	29	1
	λ_{d_v}	0.38	0.22	0.4	ρ_{d_c}	0.96	0.04
Prop	d_v	2	3	12	d_c	5	6
	# of cols	21	20	4	# of rows	30	0
	λ_{d_v}	0.28	0.4	0.32	ρ_{d_c}	1	0

For given J, L, z, $\mathbf{H}_p$, and degree distributions in $\mathbf{H}$, $\mathbf{H}_m$ should be carefully constructed for QC LDPC codes to have low error floors. For this, the submatrix of $\mathbf{H}$ consisting of the columns of low weight should have a sufficiently large girth [9], [13] and thus the construction algorithm in [9], which will be called the stepwise greedy girth-maximizing (SGGM) algorithm in this paper, is used for constructing $\mathbf{H}_m$. It is assumed that $\mathbf{H}_m$ only consists of circulant permutation matrices and zero matrices and its column weights are larger than or equal to 3.

V. SIMULATION RESULTS

To verify the effectiveness of the proposed BDD structure, the error-correcting performance of QC LDPC codes with the proposed BDD structure is compared with that with the conventional BDD structure. All simulations are performed using the belief propagation (BP) decoding under the binary AWGN channel and the maximum number of iterations is set to 100.

Four QC LDPC codes with $J = 30$, $L = 45$, and $z = 45$ are generated. The first one, denoted by 'Prop1,' has the proposed BDD structure as the parity part and its message part is constructed from the SGGM algorithm. The second one, denoted by 'Conv1,' has the conventional BDD structure as the parity part and its message part is also constructed from the SGGM algorithm. The third one, denoted by 'Prop2,' has the proposed BDD structure as the parity part and is selected among 200 QC LDPC codes with sufficiently large girth whose parity-check matrices have the message parts randomly lifted from a randomly constructed protograph. The last one, denoted by 'Conv2,' is the counterpart to 'Prop2.' These codes have the parameters $N = 2025$ and $R = 1/3$ and it is noted that the rate-$1/3$ QC LDPC codes in ETSI DVB-S2 standards [12] also have 30×45 protographs.

The degree distributions of four QC LDPC codes are shown in Table I. 'Conv1' and 'Conv2' has almost the same degree distribution as the rate-$1/3$ QC LDPC codes in ETSI DVB-S2 standards. We can see that there are too many variable nodes of degree 2 in 'Conv1' and 'Conv2' due to the conventional BDD structure. These QC LDPC codes have 1.004 as the

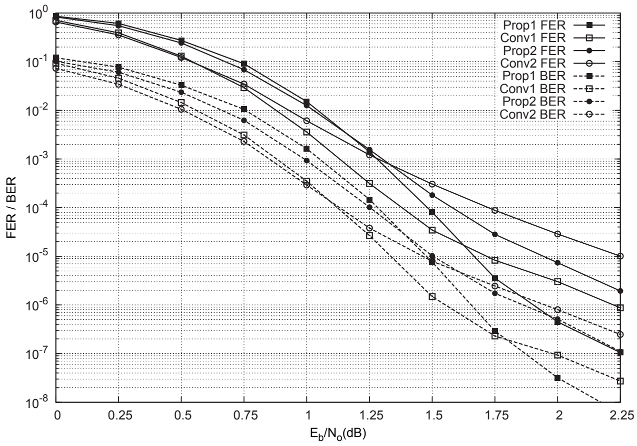

Fig. 2. FER and BER performance of QC LDPC codes with the conventional and the proposed BDD structures when $N = 2025$ and $R = 1/3$.

threshold under the AWGN channel which is calculated from [17]. Note that the girths of 'Conv1' and 'Conv2' are 8 and 6, respectively.

The degree distribution in Table I is used for the proposed BDD cases 'Prop1' and 'Prop2' with $n_2 = 9$. For fair comparison, the number of edges in the parity-check matrices of 'Prop1' and 'Prop2' is not larger than that of 'Conv1' and 'Conv2.' The threshold of these QC LDPC codes is 1.225 which is much larger than that of 'Conv1' and 'Conv2.' Note that the girths of 'Prop1' and 'Prop2' are 8 and 6, respectively.

The frame error rate (FER) and bit error rate (BER) of the above four QC LDPC codes are shown in Fig. 2. In both FER and BER, 'Prop1' and 'Prop2' show better error-correcting performance than 'Conv1' and 'Conv2' in the error floor region, respectively.

VI. Conclusions

In this paper, a new BDD structure is proposed for irregular QC LDPC codes to achieve low error floors. By adopting weight-2 circulants, the parity part of parity-check matrices can have flexible degree distribution, which can results in large minimum Hamming distance unlike the case of the conventional BDD structure. The proposed BDD structure can support the efficient encoding of QC LDPC codes by determining the set of shift values appropriately. We search for good QC LDPC codes with the proposed BDD structure by determining the ordering of nonzero shift values in the proposed BDD structure and properly locating weight-2 circulants. It is shown via numerical analysis that QC LDPC codes with the proposed BDD structure have good error-correcting performance in the error floor region compared to those with the conventional BDD structure.

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