Rate-Loss Reduction of SC-LDPC Codes by Optimizing Reliable Variable Nodes via Expected Graph Evolution

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***Abstract*—The outstanding decoding performance of spatially-coupled low-density parity-check (SC-LDPC) codes comes from wave-like propagation of reliable messages. The reliable messages are triggered by shortened (known) variable nodes in some consecutive reliable positions. However, at the cost of the improvement, shortened variable nodes cause rate-loss of SC-LDPC codes. To reduce the rate-loss, additional variable nodes (so called reliable variable nodes) can be added to the reliable positions instead of shortened variable nodes. Density evolution (DE) is an efficient method to design degree distribution of the reliable variable nodes. However, degree distributions obtained by DE show degraded performance in finite-length code performance. In this paper, we generalize the expected graph evolution and use the analysis tool in optimizing degree distribution which shows the minimum rate-loss without finite-length performance degradation. From the well-designed degree distribution, rate-loss reduction by 60% can be achieved without finite-length performance degradation.**

I. Introduction

Recently, spatially-coupled low-density parity-check (SC-LDPC) codes [1] as a new class of low-density parity-check (LDPC) codes have emerged and attracted lots of attention due to their desirable properties. Especially, SC-LDPC codes achieve the channel capacity over the general binary memoryless symmetric (BMS) channels under iterative belief propagation (BP) decoding [2].

SC-LDPC code ensemble can be constructed by shortening the variable nodes in some consecutive reliable positions of the circular SC-LDPC ensemble [1], [3]. From the shortened variable nodes, reliable messages occur and then propagate toward other variable nodes in a wave-like manner during the decoding process [4]. Due to the wave-like propagation of the reliable messages, SC-LDPC codes have improved decoding performance compared to the uncoupled LDPC codes. The improvement of SC-LDPC codes is described in terms of *threshold saturation effect*, which means that the BP threshold of SC-LDPC codes approaches to the maximum a posteriori (MAP) threshold of uncoupled LDPC codes [1], [2].

However, SC-LDPC codes suffer from rate-loss by the shortened variable nodes. The rate-loss of SC-LDPC codes can be reduced by increasing the number of component LDPC codes, but it causes other problems such as long code length and poor finite-length performance [5]. In [3] and [6], the additional variable nodes having regular degree distribution are attached by decreasing the fraction of the shortened variable nodes where the fraction is optimized by the density evolution (DE). Also, in [7], they attach additional variable nodes at the boundary of the protograph-based SC-LDPC codes using an irregular degree distribution which is optimized by DE.

However, the codes optimized by the DE show degraded finite-length performance even though their belief propagation (BP) threshold is unchanged from the conventional SC-LDPC codes. In this paper, we use expected graph evolution [5] instead of DE to optimize degree distribution of the added variable nodes without degradation in finite-length performance. To use expected graph evolution for the coupled codes having irregular variable node degree distribution, we generalize the expression of the expected graph evolution. From the experimental results, it is shown that the expected graph evolution is more appropriate to designing the codes than the DE in terms of finite-length performance.

II. Construction of Coupled Ensembles

A. Construction of SC-LDPC Ensembles with Shortened Variable Nodes

First, we consider a circular SC-LDPC ensemble [1], denoted by $\mathcal{C}_{\rm C}(l, r, L, w)$, where (l, r) are degrees of the underlying uncoupled LDPC ensemble and L and w are positive integers. The $\mathcal{C}_{\rm C}(l, r, L, w)$ ensemble consists of $L + w - 1$ positions from 1 to $L + w - 1$. In each position, there exist M variable nodes and $Ml/r \in \mathbb{N}$ check nodes. Let $\langle i \rangle$ be $(i-1 \bmod L+w-1)+1$. Each of the l edges of a variable node at position u for $u \in [1, L+w-1]$ is uniformly at random and independently connected to check nodes at positions $\langle u + i \rangle$ for $i \in [0, w - 1]$.

Let each of the positions from $L+1$ to $L+w-1$ be a reliable position. From the $\mathcal{C}_{\rm C}(l, r, L, w)$ ensemble, we can construct the SC-LDPC ensemble [1] by shortening M variable nodes at each reliable position. Further, if κ fraction of M variable nodes at each reliable position are shortened, the generalized

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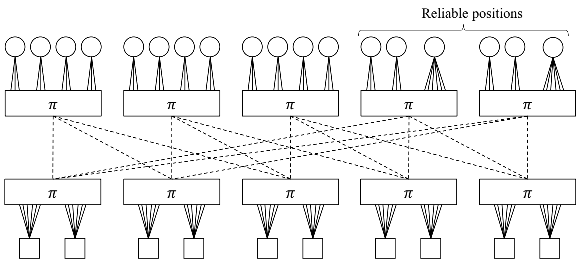

Fig. 1. Tanner graph for the $\mathcal{C}_{\mathrm{SCR}}(3,6,3,3,\frac{1}{2}x^2+\frac{1}{2}x^5)$ ensemble with $M=4$ and $M_r=3$, where the dashed edges represent a bunch of edges.

form of the SC-LDPC ensemble, $\mathcal{C}_{\mathrm{SC}}(l,r,L,w,\kappa)$ in [3], [6] can be constructed, where $0<\kappa\le 1$. The $\mathcal{C}_{\mathrm{SC}}(l,r,L,w,\kappa)$ ensemble with $\kappa=1$ corresponds to the conventional SC-LDPC ensemble in [1].

The total number of the variable nodes in the graph of the $\mathcal{C}_{\mathrm{SC}}(l,r,L,w,\kappa)$ ensemble is $V=LM+(w-1)(1-\kappa)M$. Since some of check nodes connected only with the shortened variable nodes can be removed in the graph, the expected number of check nodes in the graph is derived as [3]

$$C=\left(L+w-1-2\sum_{i=0}^{w-1}\left(\frac{\kappa i}{w}\right)^r\right)M\frac{l}{r}.$$

Then, the design rate $R_{\mathrm{SC}}=1-C/V$ of the $\mathcal{C}_{\mathrm{SC}}(l,r,L,w,\kappa)$ ensemble is given as

$$R_{\mathrm{SC}}=\left(1-\frac{l}{r}\right)-\frac{l}{r}\frac{\kappa(w-1)-2\sum_{i=0}^{w-1}(\frac{\kappa i}{w})^r}{L+(w-1)(1-\kappa)}. \qquad (1)$$

B. Construction of SC-LDPC Ensembles with Reliable Variable Nodes

We modify $\mathcal{C}_{\mathrm{SC}}(l,r,L,w,\kappa)$ ensemble to have irregular degree distribution $\lambda_{\mathrm{r}}(x)=\sum_i \lambda_{\mathrm{r},i}x^{i-1}$ [8] at the reliable positions like an approach in [7]. Let the variable nodes in each reliable position be the reliable variable nodes. The modified ensemble is called as SC-LDPC ensemble with reliable variable nodes, i.e., SCR-LDPC ensemble and denoted by $\mathcal{C}_{\mathrm{SCR}}(l,r,L,w,\lambda_{\mathrm{r}}(x))$. To construct the $\mathcal{C}_{\mathrm{SCR}}(l,r,L,w,\lambda_{\mathrm{r}}(x))$ ensemble, first remove M variable nodes at each reliable position of the $\mathcal{C}_{\mathrm{C}}(l,r,L,w)$ ensemble and then attach M_r reliable variable nodes, where M_r is the number of variable nodes at each reliable position. All Ml edges for the variable nodes in each reliable position are randomly re-distributed to the M_r reliable variable nodes following the degree distribution $\lambda_{\mathrm{r}}(x)$. Since Ml edges are connected to the M_r variable nodes in each reliable position, the following edge constraint should be satisfied as

$$M_r\frac{1}{\int_0^1\lambda_{\mathrm{r}}(x)dx}=Ml$$

where $1/\int_0^1\lambda_{\mathrm{r}}(x)dx$ is the average degree l_{avg} of the reliable variable nodes.

For example, Tanner graph for the $\mathcal{C}_{\mathrm{SCR}}(3,6,3,3,\frac{1}{2}x^2+\frac{1}{2}x^5)$ ensemble with $M=4$ is depicted in Fig. 1. The total number of variable nodes in the graph is $V=LM+(w-1)M_r$. Since all the check nodes are connected to the variable nodes, $C=(L+w-1)Ml/r$. Thus, the design rate $R_{\mathrm{SCR}}=1-C/V$ of the $\mathcal{C}_{\mathrm{SCR}}(l,r,L,w,\lambda_{\mathrm{r}}(x))$ ensemble is given as

$$\begin{aligned}R_{\mathrm{SCR}}&=\left(1-\frac{l}{r}\right)-\frac{l}{r}\frac{(w-1)\left(1-\frac{M_r}{M}\right)}{L+(w-1)\frac{M_r}{M}}\\&=\left(1-\frac{l}{r}\right)-\frac{l}{r}\frac{(w-1)\left(1-\frac{l}{l_{\mathrm{avg}}}\right)}{L+(w-1)\frac{l}{l_{\mathrm{avg}}}}.\end{aligned} \qquad (2)$$

From (2), as M_r increases, or equivalently, as the average degree l_{avg} decreases, the rate loss is reduced. Thus, optimizing $\lambda_{\mathrm{r}}(x)$ corresponds to minimizing l_{avg} without loss of decoding performance compared to the $\mathcal{C}_{\mathrm{SC}}(l,r,L,w,1)$ ensemble. In the following section, the analysis tools for predicting decoding performance are described.

III. Density Evolution and Expected Graph Evolution

A. Density Evolution of SC-LDPC and SCR-LDPC Ensembles

In this paper, the channel is assumed to be the binary erasure channel (BEC) with erasure probability ϵ. Then the DE analysis gives us asymptotic performance of a code ensemble when the code length increases infinitely. Let $x_u^{(\ell)}$ denote the average erasure probability of the outgoing messages from variable nodes at position u in iteration ℓ. Then the density evolution equations for the $\mathcal{C}_{\mathrm{SC}}(l,r,L,w,\kappa)$ ensemble are given as

$$x_u^{(\ell+1)}=\begin{cases}\epsilon\left(1-\frac{1}{w}\sum\limits_{i=0}^{w-1}\left(1-\frac{1}{w}\sum\limits_{j=0}^{w-1}x_{\langle u+i-j\rangle}^{(\ell)}\right)^{r-1}\right)^{l-1}\\ \quad\text{for } u\in[1,L]\\ \epsilon(1-\kappa)\left(1-\frac{1}{w}\sum\limits_{i=0}^{w-1}\left(1-\frac{1}{w}\sum\limits_{j=0}^{w-1}x_{\langle u+i-j\rangle}^{(\ell)}\right)^{r-1}\right)^{l-1}\\ \quad\text{for } u\in[L+1,L+w-1].\end{cases}$$

Similarly, the DE equations for the $\mathcal{C}_{\mathrm{SCR}}(l,r,L,w,\lambda_{\mathrm{r}}(x))$ ensemble are given as

$$x_u^{(\ell+1)}=\begin{cases}\epsilon\left(1-\frac{1}{w}\sum\limits_{i=0}^{w-1}\left(1-\frac{1}{w}\sum\limits_{j=0}^{w-1}x_{\langle u+i-j\rangle}^{(\ell)}\right)^{r-1}\right)^{l-1}\\ \quad\text{for } i\in[1,L]\\ \epsilon\lambda_{\mathrm{r}}\left(1-\frac{1}{w}\sum\limits_{i=0}^{w-1}\left(1-\frac{1}{w}\sum\limits_{j=0}^{w-1}x_{\langle u+i-j\rangle}^{(\ell)}\right)^{r-1}\right)\\ \quad\text{for } i\in[L+1,L+w-1].\end{cases}$$

The BP decoding thresholds of the $\mathcal{C}_{\mathrm{SC}}(l,r,L,w,\kappa)$ and $\mathcal{C}_{\mathrm{SCR}}(l,r,L,w,\lambda_{\mathrm{r}}(x))$ ensembles are defined as supremum of ϵ for which $x_i^{\ell}\to 0$ for all i as $\ell\to\infty$.

B. Expected Graph Evolution of SC-LDPC and SCR-LDPC Ensembles

When the code length is finite, the decoding performance of the $\mathcal{C}_{\mathrm{SC}}(l,r,L,w,1)$ ensemble can be predicted by the scaling law which depends on scaling parameters derived from the

$$\mathbb{E}[\Delta U_{j,u}(\tau)|\text{pos}(\tau)=m] = -j\phi_{m,u,j}(\tau) \tag{3}$$

$$\mathbb{E}[\Delta R_{j,u}(\tau)|\text{pos}(\tau)=m] =$$
$$\begin{cases} \sum_{t=2}^{\lceil \frac{l_{\max}}{w} \rceil} \xi_{m,u,t} \left(\sum_{k=1}^{t-1} kj\binom{t-1}{k}\delta_{j+1,u}^k(1-\delta_{j+1,u})^{t-1-k} - \sum_{k=1}^{t-1} kj\binom{t-1}{k}\delta_{j,u}^k(1-\delta_{j,u})^{t-1-k} \right) - 1, & \text{if } u=m, j=1 \\ \sum_{t=2}^{\lceil \frac{l_{\max}}{w} \rceil} \xi_{m,u,t} \left(\sum_{k=1}^{t-1} kj\binom{t-1}{k}\delta_{j+1,u}^k(1-\delta_{j+1,u})^{t-1-k} - \sum_{k=1}^{t-1} kj\binom{t-1}{k}\delta_{j,u}^k(1-\delta_{j,u})^{t-1-k} \right), & \text{if } u=m, j\neq 1 \\ \sum_{t=1}^{\lceil \frac{l_{\max}}{w} \rceil} \xi_{m,u,t} \left(\sum_{k=1}^{t} kj\binom{t}{k}\delta_{j+1,u}^k(1-\delta_{j+1,u})^{t-k} - \sum_{k=1}^{t} kj\binom{t}{k}\delta_{j,u}^k(1-\delta_{j,u})^{t-k} \right), & \text{if } u\neq m \end{cases} \tag{4}$$

where

$$\xi_{m,u,t}(\tau) = \sum_{i\in S(u)} \left(\sum_{k=tw}^{(t+1)w-1} \frac{w-(k \bmod w)}{w}\phi_{m,i,k}(\tau) + \sum_{k=(t-1)w+1}^{tw-1} \frac{(k \bmod w)}{w}\phi_{m,i,k}(\tau) \right) \text{ and } \delta_{j,u}(\tau) = \frac{R_{j,u}(\tau)}{\sum_{q=1}^{r} R_{q,u}(\tau)}.$$

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expected graph evolution [5]. For the expected graph evolution, the definition of $\mathcal{C}_{\text{SC}}(l,r,L,w,1)$ ensemble is modified to have the property that a variable node at position u has exactly one connection to a check node at each position $u+i, i\in[0,l-1]$ [5]. Similarly we modify the definition of the $\mathcal{C}_{\text{SC}}(l,r,L,w,\kappa)$ and $\mathcal{C}_{\text{SCR}}(l,r,L,w,\lambda_{\text{r}}(x))$ ensembles to derive the expected graph evolution. Let $\mathcal{C}_{\text{SCm}}(l,r,L,w,\kappa)$ and $\mathcal{C}_{\text{SCRm}}(l,r,L,w,\lambda_{\text{r}}(x))$ denote the modified SC-LDPC and SCR-LDPC ensembles, respectively. The construction method of the $\mathcal{C}_{\text{SCm}}(l,r,L,w,\kappa)$ and $\mathcal{C}_{\text{SCRm}}(l,r,L,w,\lambda_{\text{r}}(x))$ ensembles are described as follows.

1) Place M variable nodes of degree l at each position from 1 to L and $Ml/r\in\mathbb{N}$ check nodes of degree r at each position from 1 to $L+w-1$ on the graph. Additionally, for the $\mathcal{C}_{\text{SCm}}(l,r,L,w,\kappa)$ ensemble, place $(1-\kappa)M$ variable nodes of degree l and κM shortened variable nodes at each reliable position. Similarly, for the $\mathcal{C}_{\text{SCRm}}(l,r,L,w,\lambda_{\text{r}}(x))$ ensemble, place M_r variable nodes with degree distribution $\lambda_{\text{r}}(x)$ at each reliable position in ascending order of the degree.
2) Suppose that Ml/w is an integer. Label the variable and check node sockets from 1 to Ml at each position. Divide the Ml labeled variable node sockets at each position into w groups such that the variable node sockets with indices in $\{i+1,i+1+w,\ldots,i+1+Ml-w\}$ become group i for $i\in[0,w-1]$.
3) Let π_u be a random permutation on Ml for position u. Divide π_u into w disjoint permutations of size Ml/w, denoted by $\pi_u^0,\pi_u^1,\ldots,\pi_u^{w-1}$. Then the kth variable node socket in group i at position $\langle u-i\rangle$ is connected to the $\pi_u^i(k)$th check node socket at position u for $k\in[1,Ml/w]$, $i\in[0,w-1]$, and $u\in[1,L+w-1]$.

It is clear that the design rates of the $\mathcal{C}_{\text{SCm}}(l,r,L,w,\kappa)$ and $\mathcal{C}_{\text{SCRm}}(l,r,L,w,\lambda_{\text{r}}(x))$ ensembles are equal to those of the $\mathcal{C}_{\text{SC}}(l,r,L,w,\kappa)$ and $\mathcal{C}_{\text{SCR}}(l,r,L,w,\lambda_{\text{r}}(x))$ ensembles, respectively. Now, the expected fraction of degree-one check nodes for the $\mathcal{C}_{\text{SCm}}(l,r,L,w,\kappa)$ and $\mathcal{C}_{\text{SCRm}}(l,r,L,w,\lambda_{\text{r}}(x))$ ensembles will be derived. Although we try to follow the notations in [5], some notations are different due to the irregularity of degree distributions at the reliable positions. At iteration ℓ, let $R_{j,u}(\ell)$ be the number of edges that are connected to check nodes of degree j, $j\in[1,r]$, and located at position u, $u\in[1,L+w-1]$. Similarly, let $U_{j,u}(\ell)$ be the number of edges that are connected to variable nodes of degree j, $j\in[1,l_{\max}]$, and located at position u for $u\in[1,L+w-1]$, where $l_{\max}$ is the maximum degree of variable node in the graph. In [5], the expected initial value of $R_{j,u}(0)$ for the $\mathcal{C}_{\text{SCm}}(l,r,L,w,\kappa)$ ensemble is given as

$$\mathbb{E}[R_{j,u}(0)] = j\frac{l}{r}M\sum_{m\geq j}^{r}\rho_{m,u}\binom{m}{j}\epsilon^j(1-\epsilon)^{m-j}$$

where

$$\rho_{m,u} = \begin{cases} \binom{r}{m}\left(\frac{\kappa u}{w}\right)^m\left(1-\frac{\kappa u}{w}\right)^{r-m}, & \text{if } u\in[1,w-1] \\ 1, & \text{if } m=r, u\in[w,L] \\ 0, & \text{if } m<r, u\in[w,L] \\ \rho_{m,L+w-u}, & \text{if } u\in[L+1,L+w-1]. \end{cases}$$

The expected initial value of $U_{j,u}(0)$ for the $\mathcal{C}_{\text{SCm}}(l,r,L,w,\kappa)$ ensemble is also given as

$$\mathbb{E}[U_{j,u}(0)] = \begin{cases} \epsilon lM, & \text{if } j=l, u\in[1,L] \\ \epsilon l(1-\kappa)M, & \text{if } j=l, u\in[L+1,L+w-1] \\ 0, & \text{otherwise.} \end{cases}$$

Since all check node sockets in the $\mathcal{C}_{\text{SCRm}}(l,r,L,w,\lambda_{\text{r}}(x))$ ensemble are filled, the expected initial value of $R_{j,u}(0)$ for the $\mathcal{C}_{\text{SCRm}}(l,r,L,w,\lambda_{\text{r}}(x))$ ensemble is derived as

$$\mathbb{E}[R_{j,u}(0)] = j\frac{l}{r}M\binom{r}{j}\epsilon^r(1-\epsilon)^{r-j}.$$

Considering the distribution of the variable nodes in the $\mathcal{C}_{\text{SCRm}}(l,r,L,w,\lambda_{\text{r}}(x))$ ensemble, the expected initial value

TABLE I
THE OPTIMIZED DEGREE DISTRIBUTION OF $\lambda_{\mathrm{r}}(x) = \alpha x^{d_1} + (1-\alpha)x^{d_2}$ AND CORRESPONDING AVERAGE DEGREE

(d_1, d_2)	(3,5)	(3,6)	(3,7)	(3,8)	(4,5)	(4,6)	(4,7)	(4,8)
α	0.07	0.32	0.44	0.48	0.16	0.54	0.67	0.73
l_{avg}	4.78	4.55	4.41	4.44	4.81	4.72	4.66	4.62

of $U_{j,u}(0)$ for the $\mathcal{C}_{\mathrm{SCRm}}(l, r, L, w, \lambda_{\mathrm{r}}(x))$ ensemble is also derived as

$$\mathbb{E}[U_{j,u}(0)] = \begin{cases} \epsilon l M, & \text{if } j = l, u \in [1, L] \\ \epsilon l \lambda_{\mathrm{r},j} M, & \text{if } u \in [L+1, L+w-1] \\ 0, & \text{otherwise.} \end{cases}$$

Let $\tau = \ell/M$, $\Delta R_{j,u}(\tau) = R_{j,u}(\tau + 1/M) - R_{j,u}(\tau)$, $\Delta U_{j,u}(\tau) = U_{j,u}(\tau + 1/M) - U_{j,u}(\tau)$, and $S(i) = \{\langle i-(w-1)\rangle, \ldots, \langle i \rangle\}$. Define $\phi_{m,u,j}$ as the probability that a variable node of degree j connected to a degree-one check node at position m belongs to position u. Then, we have

$$\phi_{m,u,j}(\tau) = \begin{cases} \dfrac{U_{j,u}(\tau)}{\sum\limits_{i \in S(m)} \sum\limits_{j=1}^{l_{\max}} U_{j,i}(\tau)}, & \text{if } u \in S(m) \\ 0, & \text{otherwise .} \end{cases}$$

Let $\mathrm{pos}(\tau)$ be the position at which we remove a degree-one check node at time τ. Using $\phi_{m,u,j}(\tau)$, evolutions of $\Delta U_{j,u}(\tau)$ and $\Delta R_{j,u}(\tau)$ are derived in (3) and (4), respectively, when a degree-one check node from position m is removed. A detailed explanation for (3) and (4) is omitted due to the space limitation. From (3) and (4), the expected fraction of degree-one check nodes $\hat{r}_1(\tau)$ can be computed as

$$\mathbb{E}[\Delta R_{j,u}(\tau)] = \sum_{m=1}^{L+w-1} \mathbb{E}[\Delta R_{j,u}(\tau) | \mathrm{pos}(\tau) = m] p_m(\tau)$$

$$\mathbb{E}[\Delta U_{j,u}(\tau)] = \sum_{m=1}^{L+w-1} \mathbb{E}[\Delta U_{j,u}(\tau) | \mathrm{pos}(\tau) = m] p_m(\tau)$$

$$\hat{r}_1(\tau) = \sum_{u=1}^{L+w-1} \mathbb{E}[R_{1,u}(\tau)]/M$$

where $p_m(\tau) = R_{1,m}(\tau) / \sum_{u=1}^{L+w-1} R_{1,u}(\tau)$.

IV. Optimization for Rate-Loss Reduction

A. Optimization Based on Density Evolution

Optimizing the $\mathcal{C}_{\mathrm{SC}}(l, r, L, w, \kappa)$ ensemble for reducing the rate-loss by DE corresponds to minimizing κ without degradation of the BP threshold. From numerical result for the $\mathcal{C}_{\mathrm{SC}}(3, 6, 30, 3, \kappa)$ ensemble, we obtain the minimum κ as $\kappa_{\mathrm{opt1}} = 0.53$. From (1), the design rates of the $\mathcal{C}_{\mathrm{SC}}(3, 6, 30, 3, 1)$ and $\mathcal{C}_{\mathrm{SC}}(3, 6, 30, 3, \kappa_{\mathrm{opt1}})$ ensembles become 0.4696 and 0.4829, respectively. Thus, the rate-loss is reduced from 0.0304 to 0.0171, that is, the rate-loss is reduced by $(0.0304 - 0.0171)/0.0304 = 44\%$.

For the $\mathcal{C}_{\mathrm{SCR}}(l, r, L, w, \lambda_{\mathrm{r}}(x))$ ensemble, the optimization problem to minimize rate-loss is to minimize l_{avg} without

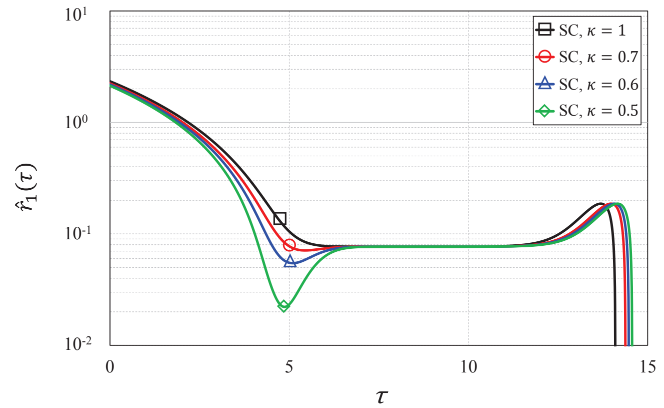

Fig. 2. $\hat{r}_1(\tau)$ for the $\mathcal{C}_{\mathrm{SCm}}(3, 6, 30, 3, \kappa)$ ensemble with various κ at $\epsilon = 0.47$.

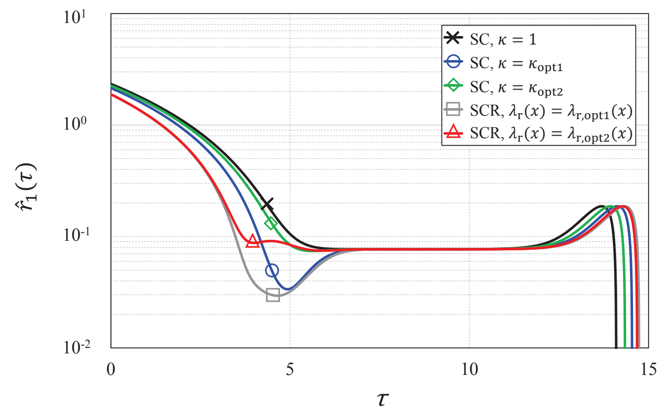

Fig. 3. $\hat{r}_1(\tau)$ for the $\mathcal{C}_{\mathrm{SCm}}(3, 6, 30, 3, \kappa)$ and $\mathcal{C}_{\mathrm{SCRm}}(3, 6, 30, 3, \lambda_{\mathrm{r}}(x))$ ensembles with $\kappa = 1$, $\kappa = \kappa_{\mathrm{opt1}}$, $\kappa = \kappa_{\mathrm{opt2}}$, $\lambda_{\mathrm{r}}(x) = \lambda_{\mathrm{r,opt1}}(x)$, and $\lambda_{\mathrm{r}}(x) = \lambda_{\mathrm{r,opt2}}(x)$ at $\epsilon = 0.47$.

degradation of the BP threshold. Solving the optimization problem is somewhat complicated because the search space of $\lambda_{\mathrm{r}}(x)$ is infinite unless we restrict the search space. However, in order to maintain regularity, which is the main characteristic of SC-LDPC codes, we use nearly regular degree distribution [9] for $\lambda_{\mathrm{r}}(x)$ such as $\lambda_{\mathrm{r}}(x) = \alpha x^{d_1 - 1} + (1-\alpha)x^{d_2 - 1}$. This nearly regular degree distribution enables us to optimize $\lambda_{\mathrm{r}}(x)$ by a brute-force search algorithm.

We restrict the maximum degree of $\lambda_{\mathrm{r}}(x)$ as 10 and the minimum degree as 3. Then, for all possible combination of (d_1, d_2), we find the optimized value α that gives the minimum average degree l_{avg} while the BP threshold is not degraded. The part of results for the $\mathcal{C}_{\mathrm{SCR}}(3, 6, 30, 3, \lambda_{\mathrm{r}}(x))$ ensemble is described in Table I, which shows that the optimized degree distribution is $\lambda_{\mathrm{r,opt1}}(x) = 0.44x^2 + 0.56x^6$. From (2), the design rate of the $\mathcal{C}_{\mathrm{SC}}(3, 6, 30, 3, \lambda_{\mathrm{r,opt1}}(x))$ ensemble is 0.4898, which gives the rate-loss reduction by 66%.

B. Optimization Based on Expected Graph Evolution

The second design rule for the optimization is based on the expected evolution $\hat{r}_1(\tau)$. In Fig. 2, $\hat{r}_1(\tau)$ for the $\mathcal{C}_{\mathrm{SCm}}(3, 6, 30, 3, \kappa)$ ensembles is depicted for various κ at

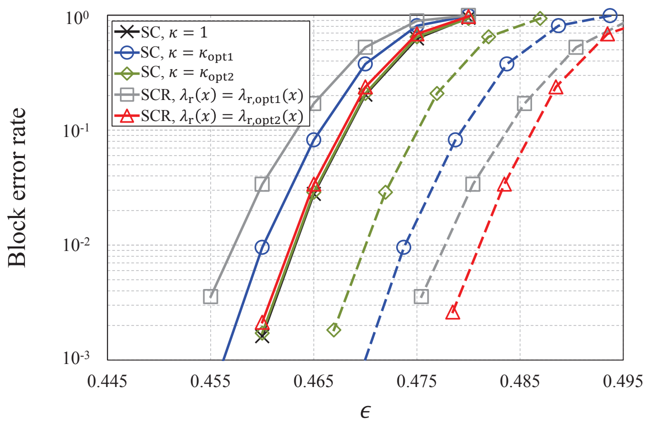

Fig. 4. Block error rate of the codes from the $\mathcal{C}_{\mathrm{SCm}}(3,6,30,3,\kappa)$ and $\mathcal{C}_{\mathrm{SCRm}}(3,6,30,3,\lambda_{\mathrm{r}}(x))$ ensembles with $\kappa=1$, $\kappa=\kappa_{\mathrm{opt1}}$, $\kappa=\kappa_{\mathrm{opt2}}$, $\lambda_{\mathrm{r}}(x)=\lambda_{\mathrm{r,opt1}}(x)$, and $\lambda_{\mathrm{r}}(x)=\lambda_{\mathrm{r,opt2}}(x)$.

$\epsilon = 0.47$. For $\kappa = 1$, there exists a critical phase [5] that is the main cause of decoding errors of SC-LDPC codes. However, as κ decreases, the other local minimum occurs before the critical phase emerges, which becomes additional cause of decoding errors so that the finite-length performance is predicted to be degraded from the case of $\kappa = 1$. Thus, the second design rule is that the expected evolution $\hat{r}_1(\tau)$ should not have the local minimum which is lower than the value in the critical phase to maintain finite-length performance. The corresponding optimization problem is to find the minimum κ or degree distribution $\lambda_{\mathrm{r}}(x)$ giving the minimum average degree while the second design rule is satisfied.

From the extensive numerical results, we obtain the optimized results such as $\kappa_{\mathrm{opt2}} = 0.74$ and $\lambda_{\mathrm{r,opt2}}(x) = 0.24x^2 + 0.76x^5$. The design rates of the $\mathcal{C}_{\mathrm{SCm}}(3,6,30,3,\kappa_{\mathrm{opt2}})$ and $\mathcal{C}_{\mathrm{SCRm}}(3,6,30,3,\lambda_{\mathrm{r,opt2}}(x))$ ensembles are 0.4762 (22% reduction) and 0.4878 (60% reduction), respectively. In Fig. 3, we show $\hat{r}_1(\tau)$ for the $\mathcal{C}_{\mathrm{SCm}}(3,6,30,3,\kappa)$ and $\mathcal{C}_{\mathrm{SCRm}}(3,6,30,3,\lambda_{\mathrm{r}}(x))$ ensembles with κ_{opt1}, κ_{opt2}, $\lambda_{\mathrm{r,opt1}}(x)$, and $\lambda_{\mathrm{r,opt2}}(x)$ at $\epsilon = 0.47$. As mentioned before, the ensembles satisfying the second design rule do not show the local minimum which is lower than the value in the critical phase.

V. Finite Length Code Performance

In this section, we show validity of the designed parameters in the previous section, κ_{opt1}, κ_{opt2}, $\lambda_{\mathrm{r,opt1}}(x)$, and $\lambda_{\mathrm{r,opt2}}(x)$, by comparing finite-length code performance. Solid lines in Fig. 4 show the block error rate of the codes from the $\mathcal{C}_{\mathrm{SCm}}(3,6,30,3,1)$, $\mathcal{C}_{\mathrm{SCm}}(3,6,30,3,\kappa_{\mathrm{opt1}})$, $\mathcal{C}_{\mathrm{SCm}}(3,6,30,3,\kappa_{\mathrm{opt2}})$, $\mathcal{C}_{\mathrm{SCRm}}(3,6,30,3,\lambda_{\mathrm{r,opt1}}(x))$, and $\mathcal{C}_{\mathrm{SCRm}}(3,6,30,3,\lambda_{\mathrm{r,opt2}}(x))$ ensembles with $M = 1000$. Note that the edge connection between variable nodes and check nodes in all positions except the reliable positions keeps the same for all codes to concentrate on the effect of newly added variable nodes in the reliable positions. The decoder uses the BP algorithm and runs the algorithm until the states of variable nodes stay unchanged. As discussed before, the code from the $\mathcal{C}_{\mathrm{SCm}}(3,6,30,3,\kappa_{\mathrm{opt2}})$ ensemble shows nearly equivalent performance with the code from the $\mathcal{C}_{\mathrm{SCm}}(3,6,30,3,1)$ ensemble. However, the rate-loss reduction of the $\mathcal{C}_{\mathrm{SCm}}(3,6,30,3,\kappa_{\mathrm{opt2}})$ ensemble is much lower than the $\mathcal{C}_{\mathrm{SCm}}(3,6,30,3,\kappa_{\mathrm{opt1}})$ ensemble derived by the first design rule. Whereas the code from the $\mathcal{C}_{\mathrm{SCRm}}(3,6,30,3,\lambda_{\mathrm{r,opt2}}(x))$ ensemble still shows large rate-loss reduction comparable to the code from the $\mathcal{C}_{\mathrm{SCRm}}(3,6,30,3,\lambda_{\mathrm{r,opt1}}(x))$ ensemble while there is no performance degradation. To clearly figure out the effect of the rate-loss reduction, we shift each of the solid lines by their rate-loss reduction to the dashed lines in Fig. 4, which shows that the SCR-LDPC ensemble with the second design rule is the most suitable codes maximizing the rate-loss reduction of SC-LDPC codes.

VI. Conclusion

In this paper, an approach to reduce rate-loss of SC-LDPC codes without degradation in finite-length code performance is proposed by using expected graph evolution. It is shown that the codes designed by the expected graph evolution show nearly equivalent decoding performance compared to conventional SC-LDPC codes while the rate-loss is significantly mitigated.

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