New Design of Low-Correlation Zone Sequence Sets

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***Abstract*—In this paper, we present several construction methods for low-correlation zone (LCZ) sequence sets. First, we propose a design scheme for binary LCZ sequence sets with parameters $(2^{n+1}-2, M, L, 2)$. In this scheme, we can freely set the LCZ length L and the resulting LCZ sequence sets have the size M, which is almost optimal with respect to Tang, Fan, and Matsufuji bound. Second, given a q-ary LCZ sequence set with parameters (N, M, L, ϵ) and even q, we construct another q-ary LCZ sequence set with parameters $(2N, 2M, L, 2\epsilon)$ or $(2N, 2M, L-1, 2\epsilon)$. Especially, the new set with parameters $(2N, 2M, L, 2)$ can be optimal in terms of the set size if a q-ary optimal LCZ sequence set with parameters $(N, M, L, 1)$ is used.**

***Index Terms*—Code-division multiple access (CDMA), low-correlation zone (LCZ) sequence, quasi-synchronous code-division multiple access (QS-CDMA), sequence.**

I. Introduction

In the code-division multiple-access (CDMA) systems, many users can share the radio resources using pseudonoise sequences with good correlation property such as the family of Gold sequences. A Gold sequence set is optimal with respect to Sidel'nikov bound [1] in the sense that the maximum magnitude of correlation achieves theoretical lower bound for a given set size and period. This lower bound is approximately equal to the square root of twice the period of sequences. Thus, even though the sequence family is optimal, the autocorrelation and cross-correlation values have relatively large magnitude. Therefore, considerable amount of multiple access interference could be introduced even though the optimal family of sequences is used.

Gaudenzi, Elia, and Vilola [2] proposed the quasi-synchronous code-division multiple-access (QS-CDMA) systems. In such systems, time delay within a few chips among different users is allowed, which gives more flexibility in designing the wireless communication systems. In the design of a sequence set for QS-CDMA systems, what matters most is to have low-correlation zone (LCZ) around the origin rather than to minimize the overall maximum nontrivial correlation value [3]. In fact, LCZ sequences with smaller correlation magnitude within the zone show better performance than other well-known sequence families with optimal correlation property [3].

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Let S be a set of M sequences of period N. If the magnitudes of correlation function between any two sequences in S take the values less than or equal to ϵ within the range, $-L < \tau < L$, of the offset τ, then S is called an (N, M, L, ϵ) LCZ sequence set. Long, Zhang, and Hu [3] proposed a binary LCZ sequence set by using Gordon–Mills–Welch (GMW) sequences [4]. For a prime p, Tang and Fan [5] proposed p-ary LCZ sequences by extending the alphabet size of each sequence in Long's work [3]. Kim, Jang, No, and Chung [6] proposed optimal quaternary LCZ sequence sets. And Jang, No, Chung, and Tang [7] constructed new optimal p-ary LCZ sequence sets. Recently, Jang, No, and Chung [8] find a new construction method of optimal p^2-ary LCZ sequence sets using unified sequences [9].

Ding, Helleseth, and Martinsen [10] proposed the new families of binary sequences with optimal three-level autocorrelation using cyclotomic number of order 4. Utilizing the interleaving technique in their work, we propose several new construction methods of LCZ sequence sets of even period.

In this paper, we present several construction methods for LCZ sequence sets. First, we propose a new design scheme for binary LCZ sequence sets with parameters $(2^{n+1}-2, M, L, 2)$. In this scheme, we can freely set the LCZ length L and the resulting LCZ sequence sets have the size M, which is almost optimal with respect to Tang, Fan, and Matsufuji bound. Second, given a q-ary LCZ sequence set with parameters (N, M, L, ϵ), we construct another q-ary LCZ sequence set with parameters $(2N, 2M, L, 2\epsilon)$ or $(2N, 2M, L-1, 2\epsilon)$. Especially if L is odd, the new set with parameters $(2N, 2M, L, 2)$ can be optimal in terms of the set size if a q-ary optimal LCZ sequence set with parameters $(N, M, L, 1)$ is used.

This paper is organized as follows. In Section II, we construct the new pool of binary sequences of period $2^{n+1}-2$ using a binary sequence of period 2^n-1 with ideal autocorrelation property and derive their correlation values. Section III proposes two methods of selecting sequences from the pool to make binary LCZ sequence sets which are almost optimal with respect to Tang, Fan, and Matsufuji bound. In Section IV, we propose two methods of obtaining new q-ary LCZ sequence sets with parameters $(2N, 2M, L, 2\epsilon)$ or $(2N, 2M, L-1, 2\epsilon)$ from a q-ary LCZ sequence set with parameters (N, M, L, ϵ) for even q.

II. Design of New Sequence Sets

Let $N = 2^{n+1}-2$. Let Z_N be the set of integers modulo N, i.e., $Z_N = \{0, 1, \ldots, N-1\}$. Let $a(t)$ be a binary sequence of period 2^n-1 with ideal autocorrelation.

Let D_u be the characteristic set of $a(t-u)$, i.e.,

$$D_u = \{t \mid a(t-u) = 1, 0 \le t \le 2^n-2\} = D_0 + u$$

where $u \in Z_{2^n-1}$, $D_0 + u = \{d + u \mid d \in D_0\}$, and "+" means addition modulo $2^n - 1$. Let $\overline{D}_u = Z_{2^n-1} \backslash D_u$. From the balancedness of $a(t)$, we have

$$|D_u| = 2^{n-1} \tag{1}$$
$$|\overline{D}_u| = 2^{n-1} - 1. \tag{2}$$

From the difference-balance property of $a(t)$, for $u \neq v$, we have

$$|D_u \cap D_v| = 2^{n-2} \tag{3}$$
$$|D_u \cap \overline{D}_v| = 2^{n-2} \tag{4}$$
$$|\overline{D}_u \cap \overline{D}_v| = 2^{n-2} - 1. \tag{5}$$

By the Chinese remainder theorem, we have $Z_N \cong Z_2 \otimes Z_{2^n-1}$ under the isomorphism $\phi : w \mapsto (w \bmod 2, w \bmod 2^n - 1)$, where $\otimes$ denotes direct product. Throughout the paper, we use the notations $w \in Z_N$ and $(w \bmod 2, w \bmod 2^n - 1)$, interchangeably.

For $u \in Z_{2^n-1}$, let C_u be the subset of Z_N such that

$$C_u \cong \{0\} \otimes A_u \cup \{1\} \otimes D_{1-u} \tag{6}$$

where A_u can be either D_u or $\overline{D}_u$. Then we have

$$|C_u| = \begin{cases} |D_u| + |D_{1-u}| = 2^n, & \text{if } A_u = D_u \\ |\overline{D}_u| + |D_{1-u}| = 2^n - 1, & \text{if } A_u = \overline{D}_u. \end{cases} \tag{7}$$

Let $s_u(t)$ be the characteristic sequence of C_u. Note that just like C_u which can be one of two distinct subsets of Z_N depending on A_u, the sequence $s_u(t)$ can also take one of two distinct sequences, one with 2^n 1's and the other with $2^n - 1$ 1's. The correlation function $R_{u,v}(\tau)$ of binary sequences $s_u(t)$ and $s_v(t)$ of period N is defined as

$$R_{u,v}(\tau) = \sum_{t=0}^{N-1} (-1)^{s_u(t)+s_v(t+\tau)}.$$

Let

$$d_{u,v}(\tau) = |C_u \cap (C_v + \tau)|$$

where $\tau \in Z_N$, $C_v + \tau = \{c + \tau \mid c \in C_v\}$, and "+" means addition modulo N. Then we can easily check the following lemma.

Lemma 1: The correlation function $R_{u,v}(\tau)$ can be expressed as

$$R_{u,v}(\tau) = N - 2(|C_u| + |C_v| - 2d_{u,v}(\tau)). \qquad \square$$

Now let us define two sets of characteristic sequences of C_u in (6).

Definition 2: The set $\mathcal{U}_1$ is the collection of all the characteristic sequences $s_u(t)$, $1 \leq u < 2^{n-1}$, of C_u with $A_u = D_u$. Similarly, the collection of all the characteristic sequences $s_u(t)$, $1 \leq u < 2^{n-1}$, of C_u with $A_u = \overline{D}_u$ is called the set $\mathcal{U}_2$. $\square$

The following theorem gives us the correlation values of the sequences in Definition 2.

Theorem 3: The correlation functions of two sequences $s_u(t)$ and $s_v(t)$ in $\mathcal{U}_1 \cup \mathcal{U}_2$ are as follows.

Case 1) $s_u(t), s_v(t) \in \mathcal{U}_1$;

i) $u \neq v$;

$$R_{u,v}(\tau) = \begin{cases} 2^n - 2, & \text{for } \tau = (0, u - v), (0, v - u), \\ & (1, u + v - 1), (1, 1 - u - v) \\ -2, & \text{otherwise.} \end{cases}$$

ii) $u = v$;

$$R_{u,u}(\tau) = \begin{cases} 2^{n+1} - 2, & \text{for } \tau = 0 \\ 2^n - 2, & \text{for } \tau = (1, 2u - 1), (1, 1 - 2u) \\ -2, & \text{otherwise.} \end{cases}$$

Case 2) $s_u(t) \in \mathcal{U}_1$ and $s_v(t) \in \mathcal{U}_2$;

i) $u \neq v$;

$$R_{u,v}(\tau) = \begin{cases} -2^n, & \text{for } \tau = (0, u - v), (1, 1 - u - v) \\ 2^n, & \text{for } \tau = (0, v - u), (1, u + v - 1) \\ 0, & \text{otherwise.} \end{cases}$$

ii) $u = v$;

$$R_{u,u}(\tau) = \begin{cases} -2^n, & \text{for } \tau = (1, 1 - 2u) \\ 2^n, & \text{for } \tau = (1, 2u - 1) \\ 0, & \text{otherwise.} \end{cases}$$

Case 3) $s_u(t), s_v(t) \in \mathcal{U}_2$;

i) $u \neq v$;

$$R_{u,v}(\tau) = \begin{cases} 2^n - 2, & \text{for } \tau = (0, u - v), (0, v - u) \\ -2^n + 2, & \text{for } \tau = (1, u + v - 1), (1, 1 - u - v) \\ -2, & \text{for } \tau = (0, \tau_2), \tau_2 \neq \pm(u - v) \\ 2, & \text{for } \tau = (1, \tau_2), \tau_2 \neq \pm(u + v - 1). \end{cases}$$

ii) $u = v$;

$$R_{u,u}(\tau) = \begin{cases} 2^{n+1} - 2, & \text{for } \tau = 0 \\ -2^n + 2, & \text{for } \tau = (1, 2u - 1), (1, 1 - 2u) \\ -2, & \text{for } \tau = (0, \tau_2), \tau_2 \neq 0 \\ 2, & \text{for } \tau = (1, \tau_2), \tau_2 \neq \pm(2u - 1). \end{cases}$$

Proof: Let $\tau = (\tau_1, \tau_2) \in Z_2 \otimes Z_{2^n-1}$. From Definition 2, it is clear that $u + v \not\equiv 1 \bmod 2^n - 1$. Then we have

$$\begin{aligned} d_{u,v}(\tau) &= |C_u \cap (C_v + \tau)| \\ &= |\{0\} \cap \{\tau_1\}||A_u \cap (A_v + \tau_2)| \\ &\quad + |\{0\} \cap \{1 + \tau_1\}||A_u \cap (D_{1-v} + \tau_2)| \\ &\quad + |\{1\} \cap \{\tau_1\}||D_{1-u} \cap (A_v + \tau_2)| \\ &\quad + |\{1\} \cap \{1 + \tau_1\}||D_{1-u} \cap (D_{1-v} + \tau_2)| \\ &= \begin{cases} |A_u \cap (A_v + \tau_2)| + |D_{1-u} \cap (D_{1-v} + \tau_2)|, & \text{for } \tau_1 = 0 \\ |A_u \cap (D_{1-v} + \tau_2)| + |D_{1-u} \cap (A_v + \tau_2)|, & \text{for } \tau_1 = 1. \end{cases} \end{aligned} \tag{8}$$

Case 1) $s_u(t), s_v(t) \in \mathcal{U}_1$;

In this case, we have $A_u = D_u$ and $A_v = D_v$.

i) $u \neq v$;

From (8), we have (9) shown at the bottom of the page. Applying (1) and (3) to (9), we have

$$d_{u,v}(\tau)=\begin{cases}2^{n-1}+2^{n-2}, & \text{for } \tau=(0,u-v),(0,v-u),\\ & (1,u+v-1),(1,1-u-v)\\ 2^{n-1}, & \text{otherwise}.\end{cases}$$

From Lemma 1 and (7), we have

$$R_{u,v}(\tau)=\begin{cases}2^{n}-2, & \text{for } \tau=(0,u-v),(0,v-u),\\ & (1,u+v-1),(1,1-u-v)\\ -2, & \text{otherwise}.\end{cases}$$

ii) $u=v$;

This case corresponds to autocorrelation of the sequences in $\mathcal{U}_1$ and we have

$$\begin{aligned}d_{u,u}(\tau)&\\ =&\begin{cases}|D_u\cap|D_u+\tau_2|+|D_{1-u}\cap(D_{1-u}+\tau_2)|, & \text{for } \tau_1{=}0\\ |D_u\cap|D_{1-u}+\tau_2|+|D_{1-u}\cap(D_u+\tau_2)|, & \text{for } \tau_1{=}1\end{cases}\\ =&\begin{cases}2^{n}, & \text{for } \tau=0\\ 2^{n-1}+2^{n-2}, & \text{for } \tau=(1,2u-1),(1,1-2u)\\ 2^{n-1}, & \text{otherwise}.\end{cases}\end{aligned}$$

Then we have

$$R_{u,u}(\tau)=\begin{cases}2^{n+1}-2, & \text{for } \tau=0\\ 2^{n}-2, & \text{for } \tau=(1,2u-1),(1,1-2u)\\ -2, & \text{otherwise}.\end{cases}\tag{10}$$

Case 2) $s_u(t)\in\mathcal{U}_1$ and $s_v(t)\in\mathcal{U}_2$;

In this case, we have $A_u=D_u$ and $A_v=\overline{D}_v$.

i) $u\neq v$;

From (8), we have the second equation shown at the bottom of the page. Using (1)–(4), we have

$$\begin{aligned}d_{u,v}(\tau)&\\ =&\begin{cases}2^{n-2}+0, & \text{for } \tau=(0,u-v),(1,1-u-v)\\ 2^{n-2}+2^{n-1}, & \text{for } \tau=(0,v-u),(1,u+v-1)\\ 2^{n-1}, & \text{otherwise}\end{cases}\end{aligned}$$

which yields

$$R_{u,v}(\tau)=\begin{cases}-2^{n}, & \text{for } \tau=(0,u-v),(1,1-u-v)\\ 2^{n}, & \text{for } \tau=(0,v-u),(1,u+v-1)\\ 0, & \text{otherwise}.\end{cases}$$

ii) $u=v$;

We have

$$R_{u,u}(\tau)=\begin{cases}-2^{n}, & \text{for } \tau=(1,1-2u)\\ 2^{n}, & \text{for } \tau=(1,2u-1)\\ 0, & \text{otherwise}.\end{cases}$$

Case 3) $s_u(t)$, $s_v(t)\in\mathcal{U}_2$;

In this case, we have $A_u=\overline{D}_u$ and $A_v=\overline{D}_v$.

i) $u\neq v$;

From (8), we have the third equation shown at the bottom of the page. Using (1)–(5) and $|D_u\cap\overline{D}_u|=0$, we have

$$\begin{aligned}d_{u,v}(\tau)&\\ =&\begin{cases}2^{n-1}+2^{n-2}-1, & \text{for } \tau=(0,u-v),(0,v-u)\\ 2^{n-2}, & \text{for } \tau=(1,u+v-1),\\ & (1,1-u-v)\\ 2^{n-1}-1, & \text{for } \tau=(0,\tau_2),\tau_2\neq\pm(u-v)\\ 2^{n-1}, & \text{for } \tau=(1,\tau_2),\\ & \tau_2\neq\pm(u+v-1)\end{cases}\end{aligned}$$

$$d_{u,v}(\tau)=\begin{cases}|D_u|+|D_{1-u}\cap D_{u-2v+1}|, & \text{for } \tau=(0,u-v)\\ |D_u\cap D_{2v-u}|+|D_{1-u}|, & \text{for } \tau=(0,v-u)\\ |D_u\cap(D_v+\tau_2)|+|D_{1-u}\cap(D_{1-v}+\tau_2)|, & \text{for } \tau=(0,\tau_2),\tau_2\neq\pm(u-v)\\ |D_u|+|D_{1-u}\cap D_{u+2v-1}|, & \text{for } \tau=(1,u+v-1)\\ |D_u\cap D_{2-u-2v}|+|D_{1-u}|, & \text{for } \tau=(1,1-u-v)\\ |D_u\cap(D_{1-v}+\tau_2)|+|D_{1-u}\cap(D_v+\tau_2)|, & \text{for } \tau=(1,\tau_2),\tau_2\neq\pm(u+v-1).\end{cases}\tag{9}$$

$$d_{u,v}(\tau)=\begin{cases}|D_u\cap\overline{D}_u|+|D_{1-u}\cap D_{u-2v+1}|, & \text{for } \tau=(0,u-v)\\ |D_u\cap\overline{D}_{2v-u}|+|D_{1-u}|, & \text{for } \tau=(0,v-u)\\ |D_u\cap(\overline{D}_v+\tau_2)|+|D_{1-u}\cap(D_{1-v}+\tau_2)|, & \text{for } \tau=(0,\tau_2),\tau_2\neq\pm(u-v)\\ |D_u|+|D_{1-u}\cap\overline{D}_{u+2v-1}|, & \text{for } \tau=(1,u+v-1)\\ |D_u\cap D_{2-u-2v}|+|D_{1-u}\cap\overline{D}_{1-u}|, & \text{for } \tau=(1,1-u-v)\\ |D_u\cap(D_{1-v}+\tau_2)|+|D_{1-u}\cap(\overline{D}_v+\tau_2)|, & \text{for } \tau=(1,\tau_2),\tau_2\neq\pm(u+v-1).\end{cases}$$

$$d_{u,v}(\tau)=\begin{cases}|\overline{D}_u|+|D_{1-u}\cap D_{u-2v+1}|, & \text{for } \tau=(0,u-v)\\ |\overline{D}_u\cap\overline{D}_{2v-u}|+|D_{1-u}|, & \text{for } \tau=(0,v-u)\\ |\overline{D}_u\cap(\overline{D}_v+\tau_2)|+|D_{1-u}\cap(D_{1-v}+\tau_2)|, & \text{for } \tau=(0,\tau_2),\tau_2\neq\pm(u-v)\\ \overline{D}_u\cap D_u|+|D_{1-u}\cap\overline{D}_{u+2v-1}|, & \text{for } \tau=(1,u+v-1)\\ |\overline{D}_u\cap D_{2-u-2v}|+|D_{1-u}\cap\overline{D}_{1-u}|, & \text{for } \tau=(1,1-u-v)\\ |\overline{D}_u\cap(D_{1-v}+\tau_2)|+|D_{1-u}\cap(\overline{D}_v+\tau_2)|, & \text{for } \tau=(1,\tau_2),\tau_2\neq\pm(u+v-1).\end{cases}$$

which gives us

$$R_{u,v}(\tau) = \begin{cases} 2^n - 2, & \text{for } \tau = (0, u-v), (0, v-u) \\ -2^n + 2, & \text{for } \tau = (1, u+v-1), (1, 1-u-v) \\ -2, & \text{for } \tau = (0, \tau_2), \tau_2 \neq \pm(u-v) \\ 2, & \text{for } \tau = (1, \tau_2), \tau_2 \neq \pm(u+v-1). \end{cases}$$

ii) $u = v$;
Similarly to (10), the autocorrelation function of the sequences in $\mathcal{U}_2$ can be obtained as

$$R_{u,u}(\tau) = \begin{cases} 2^{n+1} - 2, & \text{for } \tau = 0 \\ -2^n + 2, & \text{for } \tau = (1, 2u-1), (1, 1-2u) \\ -2, & \text{for } \tau = (0, \tau_2), \tau_2 \neq 0 \\ 2, & \text{for } \tau = (1, \tau_2), \tau_2 \neq \pm(2u-1). \end{cases}$$

□

Note that Case 1)-ii) and Case 3)-ii) correspond to autocorrelation functions. Also note that there are two sidelobes, i.e., correlation magnitude exceeding $\epsilon = 2$, for each correlation function with $u = v$, and four sidelobes, otherwise. The following example shows the construction procedure.

Example 4: For $n = 4$, we have the binary m-sequence of period 15 given by

$$000100110101111.$$

Then the support set of this m-sequence is given as

$$D_0 = \{3, 6, 7, 9, 11, 12, 13, 14\}.$$

For $u = 6$, we have

$$D_6 = \{0, 2, 3, 4, 5, 9, 12, 13\}$$

and

$$D_{-5} = \{1, 2, 4, 6, 7, 8, 9, 13\}.$$

Thus the sequence $s_6(t) \in \mathcal{U}_1$ with the characteristic set

$$\begin{aligned} C_6 &= \{0\} \otimes D_6 \cup \{1\} \otimes D_{-5} \\ &= \{0, 1, 2, 4, 7, 9, 12, 13, 17, 18, 19, 20, 21, 23, 24, 28\} \end{aligned}$$

is given as

$$s_6(t) = 111010010100110001111101100010.$$

Similarly, for $v = 3$, we can obtain the sequence $s_3(t) \in \mathcal{U}_1$ as

$$s_3(t) = 111001110111101010010000110100$$

and for $v = 4$, $s_4(t) \in \mathcal{U}_1$ is given as

$$s_4(t) = 101100000111000110110111011010.$$

The cross-correlation values of $s_6(t)$ and $s_3(t)$ are listed as

$$\begin{aligned} R_{6,3}(\tau) = &-2, -2, -2, -2, -2, -2, -2, 14, -2, -2, -2, -2, \\ &14, -2, -2, -2, -2, -2, 14, -2, -2, -2, -2, 14, \\ &-2, -2, -2, -2, -2, -2. \end{aligned}$$

In this case, LCZ is 7 with $\epsilon = 2$. Similarly, the cross-correlation values of $s_6(t)$ and $s_4(t)$ are listed as

$$\begin{aligned} R_{6,4}(\tau) = &-2, -2, 14, -2, -2, -2, -2, -2, -2, 14, -2, -2, \\ &-2, -2, -2, -2, -2, -2, -2, -2, -2, 14, -2, -2, \\ &-2, -2, -2, -2, 14, -2. \end{aligned}$$

In this case, LCZ is 2 with $\epsilon = 2$. □

As one can see in the example, there exist various LCZs in correlation functions between sequences in $\mathcal{U}_1 \cup \mathcal{U}_2$. In the next section, we design LCZ sequence sets by selecting appropriate sequences in $\mathcal{U}_1 \cup \mathcal{U}_2$.

III. Constructions of Almost Optimal Binary LCZ Sequence Sets

In this section, we will propose two methods of selecting binary sequences in $\mathcal{U}_1 \cup \mathcal{U}_2$, so that the set consisting of the selected sequences forms a binary LCZ sequence set which is almost optimal with respect to the following bound.

Theorem 5: (Tang, Fan, and Matsufuji [11]) Let $\mathcal{S}$ be an LCZ sequence set with parameters (N, M, L, ϵ). Then we have

$$ML - 1 \leq \frac{N-1}{1 - \epsilon^2/N}. \tag{11}$$

□

Since $\epsilon = 2$ in our case, (11) becomes

$$ML \leq N + 4 + \frac{12}{N-4}$$

and for $n \geq 4$, we have

$$M \leq \left\lfloor \frac{N+4}{L} \right\rfloor \tag{12}$$

where $\lfloor x \rfloor$ means the greatest integer less than or equal to x. When an $(N, M, L, 2)$ LCZ sequence set achieves the equality in (12), it is said to be optimal.

Recall that the locations of sidelobes are symmetric with respect to the origin. Thus, in terms of the distances to the sidelobes from the origin, there are at most two distinct distances. Let $L_{u,v}$ denote the distance to the nearest sidelobes from the origin in $R_{u,v}(\tau)$. Then, $L_{u,v}$ can be determined as in the following lemma.

Lemma 6: For $s_u(t), s_v(t) \in \mathcal{U}_1 \cup \mathcal{U}_2$, $1 \leq v \leq u < 2^{n-1}$, $L_{u,v}$ is given as

$$L_{u,v} = \begin{cases} \frac{N}{2} - u - v + 1, & \text{if } u - v \text{ is odd} \\ u - v, & \text{if } u - v \text{ is even and } u \neq v \\ 2u - 1, & \text{if } u = v. \end{cases} \tag{13}$$

Proof: It is not difficult to see that the distance d_1 from 0 to $(0, u-v)$ is

$$d_1 = \begin{cases} u-v, & \text{if } u-v \text{ is even} \\ \frac{N}{2}-(u-v), & \text{if } u-v \text{ is odd.} \end{cases}$$

Similarly, the distance d_2 from 0 to $(1, u+v-1)$ is

$$d_2 = \begin{cases} u+v-1, & \text{if } u-v \text{ is even} \\ \frac{N}{2}-(u+v-1), & \text{if } u-v \text{ is odd.} \end{cases}$$

Thus, $L_{u,v}$, the minimum of d_1, d_2, and $2u-1$, can be easily deduced as in (13). □

Lemma 6 tells us that the LCZ of a set of sequences $s_u(t)$'s chosen from $\mathcal{U}_1 \cup \mathcal{U}_2$ is solely dependent on the index values u's regardless of whether the sequence $s_u(t)$ is from $\mathcal{U}_1$ or $\mathcal{U}_2$.

Thus what we are going to do now is to choose an index set $I \subset \{1, 2, \ldots, 2^{n-1}-1\}$ and to construct the set of sequences

$$W_I = \{s_u(t) \in \mathcal{U}_1 \mid u \in I\} \cup \{s_u(t) \in \mathcal{U}_2 \mid u \in I\}$$

so that W_I becomes a good LCZ sequence set.

Lemma 6 tells us that the LCZ of the set W_I is the minimum of the following three values: $2^n-(u+v)$ for odd $|u-v|$, $|u-v|$ for nonzero even $|u-v|$, and $2u-1$ for $u=v$ as u and v run over I.

Thus, to maintain a given parameter L, the LCZ of the set W_I,

i) the indices in I should be greater than or equal to $\frac{(L+1)}{2}$;
ii) sum of two indices should be less than or equal to 2^n-L unless their differences are even;
iii) their differences should not be even numbers less than L.

At the same time, for a given L, we want to make the size of I as large as possible.

From these constraints, we can formulate fairly complex optimal design problem. The solution for this problem seems somewhat complicated, but aforementioned constraints implicitly lead us to consider an index set I which forms an arithmetic progression with odd value of common difference.

Construction 1: Pick an odd integer f and a nonnegative integer $f_0 < f$. Then we make an index set I as

$$I = \left\{ f_0 + mf \;\middle|\; m = 1, 2, \ldots, \left\lfloor \frac{2^{n-1}-f_0}{f} \right\rfloor \right\}. \qquad □$$

Then it is not difficult to show that the set size M and LCZ L of W_I in Construction 1 are given as in the following theorem.

Theorem 7: Let q and r be the quotient and the remainder of 2^{n-1}, respectively, when divided by f, i.e., $2^{n-1} = qf + r$. Then W_I from Construction 1 becomes a binary LCZ sequence set with parameters $(2^{n+1}-2, M, L, 2)$, where M and L are given as

$$M = 2q$$

and if $f_0 = 0$,

$$L = \begin{cases} f+2r, & \text{for } f \geq 2r+1 \\ 2f-1, & \text{for } f < 2r+1 \end{cases} \tag{14}$$

and if $f_0 \neq 0$,

$$L = \begin{cases} f+2r-2f_0, & \text{for } f \geq 2r-2f_0 \\ 2f, & \text{for } f < 2r-2f_0. \end{cases} \tag{15}$$

Proof: From Lemma 6 and the fact that f is odd, L is the smallest value of $2f$, $2f+2f_0-1$, and $2^n-(u+v)$, where u and v are the largest and the second largest elements in I. Since $u+v = 2qf - f + 2f_0$, we have $2^n-(u+v) = f+2r-2f_0$. Therefore, we have

$$L = \min\{2f, 2f+2f_0-1, f+2r-2f_0\}. \tag{16}$$

We can obtain (14) and (15) from (16). □

Note that if f is even, then from Lemma 6, LCZ of the sequence set W_I becomes

$$L = \min_{u,v \in I, u \neq v} (u-v) = f.$$

But if f is odd, then from Theorem 7, LCZ is greater than f, which is the reason why we make the common difference f odd.

Now, we can easily obtain the following corollary.

Corollary 8: The product of set size and LCZ in Construction 1 is given in (17) shown at the bottom of the page.

We check the optimality of the sets from Construction 1 and their parameters are listed in Table I. By comparing the result in (17) with the bound in (12), one can easily see that our construction corresponding to the first three cases of the inequality (17) cannot be optimal. But, as one can see in Table I, most of the sets are almost optimal, although we cannot find any which is optimal. This observation motivates us to consider a little modification to the index set I. In the following construction method, we allow two distinct values $f+2$ and f for the difference values between adjacent indices in I.

Construction 2: The indices u of the selected sequences $s_u(t)$ both in $\mathcal{U}_1$ and in $\mathcal{U}_2$ are chosen to form a progression starting from $f+2-f_0$ with differences f and $f+2$, alternately, i.e.,

$$I = \{u_j \mid j = 0, 1, 2, \ldots, J, \; u_0 = f+2-f_0, \\ u_{2k+1} - u_{2k} = f, \; u_{2k+2} - u_{2k+1} = f+2\}$$

$$ML = \begin{cases} N - M(f-2r) - 4r + 2, & \text{for } f \geq 2r+1 \text{ and } f_0 = 0 \\ N - M - 4r + 2, & \text{for } f < 2r+1 \text{ and } f_0 = 0 \\ N - M(f-2r+2f_0) - 4r + 2, & \text{for } f \geq 2(r-f_0) \text{ and } f_0 \neq 0 \\ N - 4r + 2, & \text{for } f < 2(r-f_0) \text{ and } f_0 \neq 0. \end{cases} \tag{17}$$

□

TABLE I
POSSIBLE CONSTRUCTIONS OF LCZ SEQUENCE SETS OF PERIOD $N = 2^{n+1} - 2$

Construction 1						Construction 2					
N	f	f_0	L	M	M^*	N	f	f_0	L	M	M^*
30	3	0	5	4	6	30	1	0	4	6	8
30	5	0	9	2	3						
62	3	0	5	10	13	62	1	0	4	14	16
62	5	0	7	6	9	62	3	0	8	6	8
62	7	0	11	4	6	62	5	0	12	4	5
62	11	0	21	2	3						
126	3	0	5	20	26	126	1	0	4	30	32
126	5	0	9	12	14	126	3	0	8	14	16
126	7	0	13	8	10	126	5	1	11	10	11
126	9	0	17	6	7	126	7	0	16	6	8
126	13	0	25	4	5	126	11	0	24	4	5
254	3	0	5	42	51	254	1	0	4	62	64
254	5	0	9	24	28	254	3	0	8	30	32
254	9	0	11	14	23	254	5	0	12	20	21
254	11	1	22	10	11	254	7	0	16	14	16
254	13	1	26	8	9	254	9	1	19	12	13
254	17	1	34	6	7	254	11	1	21	10	12
254	25	1	50	4	5	254	13	0	28	8	9
254	43	0	85	2	3	254	17	0	36	6	7
510	3	0	5	84	102	510	1	0	4	126	128
510	5	0	9	50	57	510	3	0	8	62	64
510	7	0	11	36	46	510	5	1	11	42	46
510	9	0	13	28	39	510	7	0	16	30	32
510	11	1	22	22	23	510	9	0	20	24	25
510	13	1	26	18	19	510	11	0	24	20	21
510	15	0	29	16	17	510	13	1	19	18	27
510	17	0	33	14	15	510	15	0	32	14	16
510	19	1	38	12	13	510	19	1	37	12	13
510	23	1	46	10	11	510	21	0	44	10	11
510	27	1	54	8	9	510	27	0	56	8	9
510	37	0	71	6	7	510	35	0	72	6	7
510	51	0	101	4	5	510	51	1	101	4	5
1022	3	0	5	170	205	1022	1	0	4	254	256
1022	5	0	7	102	146	1022	3	0	8	126	128
1022	7	0	13	72	78	1022	5	0	12	84	85
1022	9	0	17	56	60	1022	7	0	16	62	64
1022	11	0	17	46	60	1022	9	0	20	50	51
1022	13	1	26	38	39	1022	11	1	21	42	48
1022	15	0	17	34	60	1022	13	1	23	36	44
1022	17	0	19	30	54	1022	15	0	32	30	32
1022	19	0	37	26	27	1022	17	1	27	28	38
1022	21	0	29	24	35	1022	19	0	40	24	25
1022	23	0	29	22	35	1022	21	0	44	22	23
1022	25	0	37	20	27	1022	23	0	48	20	21
1022	27	0	53	18	19	1022	25	0	52	18	19
1022	29	1	58	16	17	1022	29	0	60	16	17
1022	33	1	66	14	15	1022	33	0	68	14	15
1022	39	1	78	12	13	1022	37	0	76	12	13
1022	45	1	90	10	11	1022	45	1	91	10	11
1022	57	0	113	8	9	1022	55	0	112	8	9
1022	73	0	145	6	7	1022	71	0	144	6	7
1022	103	0	203	4	5	1022	101	0	204	4	5
2046	3	0	5	340	410	2046	1	0	4	510	512
2046	5	0	9	204	227	2046	3	0	8	254	256
2046	7	0	9	146	227	2046	5	1	11	170	186
2046	9	1	18	112	113	2046	7	0	16	126	128
2046	11	0	21	92	97	2046	9	1	15	102	136
2046	13	0	23	78	89	2046	11	0	24	84	85
2046	15	0	19	68	107	2046	13	0	28	72	73
2046	17	0	21	60	97	2046	15	0	32	62	64
2046	19	1	38	52	53	2046	17	1	35	56	58
2046	21	0	37	48	55	2046	19	0	40	50	51
2046	23	0	35	44	58	2046	21	1	35	46	58
2046	25	0	49	40	41	2046	23	1	41	42	50
2046	27	1	54	36	37	2046	25	0	52	38	39
2046	29	1	58	34	35	2046	27	1	45	36	45
2046	31	0	61	32	33	2046	29	1	35	34	58
2046	33	0	65	30	31	2046	31	0	64	30	32

M^* means the maximum set size for a given L obtained from the Tang, Fan, and Matsufuji bound.

where J is the largest integer such that $u_J < 2^{n-1}$, f_0 is 0 or 1, and f is some odd integer. □

The set size M and LCZ L are given as in the following theorem.

Theorem 9: Let q and r be the quotient and the remainder of $2^{n-1}-1$, respectively, when divided by $2(f+1)$, i.e., $2^{n-1}-1 = 2q(f+1)+r$. Then W_I from Construction 2 becomes a binary LCZ sequence set with parameters $(2^{n+1}-2, M, L, 2)$, where M and L are given as

$$M = \begin{cases} 4q, & \text{for } 0 \le r < f+2-f_0 \\ 4q+2, & \text{for } f+2-f_0 \le r < 2f+2 \end{cases}$$

and

$$L = \begin{cases} 2r+f+2+2f_0, & \text{for } 0 \le r < \frac{f-3f_0}{2} \\ 2f+2-f_0, & \text{for } \frac{f-3f_0}{2} \le r < f+2-f_0 \text{ and } \\ & \frac{3f+2-3f_0}{2} \le r < 2f+2 \\ 2r-f+2f_0, & \text{for } f+2-f_0 \le r < \frac{3f+2-3f_0}{2}. \end{cases} \tag{18}$$

Proof: From Lemma 6 and the fact that f is odd, L is the smallest value of $2f+2$, $2(f+2-f_0)-1$, and $2^n-(u+v)$, where u and v are the largest and the second largest elements in I.

If $0 \le r < f+2-f_0$, then $|I| = 2q$ and $u+v = 4q(f+1)-f-2f_0$. Thus, $2^n-(u+v) = f+2+2r+2f_0$. Therefore, we have

$$L = \min\{2f+2, 2f+3-2f_0, f+2+2r+2f_0\}. \tag{19}$$

If $f+2-f_0 \le r < 2f+2$, then $|I| = 2q+1$ and $u+v = 4q(f+1)+f+2-2f_0$. Thus $2^n-(u+v) = 2r-f+2f_0$. Therefore, we have

$$L = \min\{2f+2, 2f+3-2f_0, 2r-f+2f_0\}. \tag{20}$$

We can obtain (18) from (19) and (20). □

Now, we can easily obtain the following corollary.

Corollary 10: See the equation at the bottom of the page.

Table I lists the parameters M and L of the binary LCZ sequence sets with parameters $(2^{n+1}-2, M, L, 2)$ constructed from Constructions 1 and 2. Note that the sets from Constructions 1 and 2 have many different L and M for the same period. Table I shows the flexibility in the construction of LCZ sequence sets and a tradeoff between the set size and the LCZ of the set. We believe that our design scheme is useful in the sense that it can provide flexibility to the design of LCZ sequence sets, which are almost optimal.

IV. Extension of LCZ Sequence Sets

In this section, we discuss about the method of obtaining an extended LCZ sequence set from a given LCZ sequence set. Here the term "extended" implies either lengthening the period or enlarging the set size or both.

We are proposing two methods of obtaining an LCZ sequence set having twice the period as well as twice the set size from a given set. The first method can be applied to a q-ary (N, M, L, ϵ) LCZ sequence set with an even integer q. The extended LCZ sequence set has the parameters $(2N, 2M, L, 2\epsilon)$ or $(2N, 2M, L-1, 2\epsilon)$, depending on the parity of L.

Let q be an even integer. Let $\mathcal{L}_1$ be a q-ary LCZ sequence set with parameters (N, M, L, ϵ) given as

$$\mathcal{L}_1 = \{v_i(t) \mid 0 \le i \le M-1,\ 0 \le t \le N-1\}.$$

We will call $\mathcal{L}_1$ the component LCZ sequence set. Using the component LCZ sequence set, we can construct a new LCZ sequence set with twice the size and period.

Construction 3: Let $\mathcal{T}_1$ be the set of q-ary sequences given as

$$\mathcal{T}_1 = \{s_i(t) \mid 0 \le i \le 2M-1,\ 0 \le t \le 2N-1\}$$

where $s_i(t)$ is defined as

$$s_i(2t) = \begin{cases} v_i(t), & \text{for } 0 \le i \le M-1 \\ v_{i-M}(t) + \frac{q}{2} & \text{for } M \le i \le 2M-1 \end{cases}$$

$$s_i(2t+1) = \begin{cases} v_i\left(t + \left\lceil \frac{L}{2} \right\rceil\right) & \text{for } 0 \le i \le M-1 \\ v_{i-M}\left(t + \left\lceil \frac{L}{2} \right\rceil\right) & \text{for } M \le i \le 2M-1 \end{cases}$$

where $\lceil x \rceil$ means the smallest integer greater than or equal to x. □

Theorem 11: $\mathcal{T}_1$ in Construction 3 is a q-ary LCZ sequence set with parameters $(2N, 2M, L, 2\epsilon)$ if L is odd and with parameters $(2N, 2M, L-1, 2\epsilon)$ if L is even.

Proof: We will prove the case when L is odd. The case of even L can be proven similarly. Let $R_{i,j}(\tau)$ be the correlation between $s_i(t)$ and $s_j(t)$ given as

$$R_{i,j}(\tau) = \sum_{t=0}^{2N-1} \omega^{s_i(t)-s_j(t+\tau)}$$

where ω is a complex qth root of unity. Then we must consider the following six cases.

Case 1) $0 \le i, j \le M-1$ and τ is even.

In this case, $R_{i,j}(\tau)$ can be rewritten as

$$R_{i,j}(\tau) = \sum_{t=0}^{N-1} \omega^{v_i(t)-v_j(t+\frac{\tau}{2})} + \sum_{t=0}^{N-1} \omega^{v_i(t+\frac{L+1}{2})-v_j(t+\frac{L+1}{2}+\frac{\tau}{2})}. \tag{21}$$

$$ML = \begin{cases} N-M(f-2r-2f_0)-4r-2, & \text{for } 0 \le r < \frac{f-3f_0}{2} \\ N-Mf_0-4r-2, & \text{for } \frac{f-3f_0}{2} \le r < f+2-f_0 \\ N-M(3f-2r-2f_0+2)-4(r-f)+2 & \text{for } f+2-f_0 \le r < \frac{3f+2-3f_0}{2} \\ N-Mf_0+2-4(r-f) & \text{for } \frac{3f+2-3f_0}{2} \le r < 2f+2. \end{cases}$$ □

From the property of the LCZ sequence set with parameters (N, M, L, ϵ), it is clear that the magnitude of each summation in (21) is less than or equal to ϵ within the range $-2L < \tau < 2L$. Therefore, $|R_{i,j}(\tau)| \le 2\epsilon$ within the range $-2L < \tau < 2L$.

Case 2) $0 \le i, j \le M-1$ and τ is odd.

In this case, $R_{i,j}(\tau)$ can be rewritten as

$$R_{i,j}(\tau) = \sum_{t=0}^{N-1} \omega^{v_i(t) - v_j(t + \frac{L+1}{2} + \frac{\tau-1}{2})} + \sum_{t=0}^{N-1} \omega^{v_i(t+\frac{L+1}{2}) - v_j(t + \frac{\tau+1}{2})}. \tag{22}$$

From the property of the LCZ sequence set with parameters (N, M, L, ϵ) and in-phase autocorrelation, it is clear that the magnitude of the first summation in (22) is less than or equal to ϵ within the range $-3L < \tau < L$. And it is also clear that the magnitude of the second summation in (22) is less than or equal to ϵ within the range $-L < \tau < 3L$. Therefore, $|R_{i,j}(\tau)| \le 2\epsilon$ within the range $-L < \tau < L$.

Case 3) $0 \le i \le M-1$, $M \le j \le 2M-1$ (or $M \le i \le 2M-1$, $0 \le j \le M-1$), and τ is even.

It is easy to see that $\omega^{-q/2} = -1$. Therefore, $R_{i,j}(\tau)$ can be rewritten as

$$\begin{aligned} R_{i,j}(\tau) &= \sum_{t=0}^{N-1} \omega^{v_i(t) - v_{j-M}(t+\frac{\tau}{2})} + \sum_{t=0}^{N-1} \omega^{v_i(t+\frac{L+1}{2}) - v_{j-M}(t+\frac{L+1}{2}+\frac{\tau}{2}) - \frac{q}{2}} \\ &= \sum_{t=0}^{N-1} \omega^{v_i(t) - v_{j-M}(t+\frac{\tau}{2})} - \sum_{t=0}^{N-1} \omega^{v_i(t+\frac{L+1}{2}) - v_{j-M}(t+\frac{L+1}{2}+\frac{\tau}{2})}. \end{aligned}$$

Similarly to Case 1), $|R_{i,j}(\tau)| \le 2\epsilon$ within the range $-2L < \tau < 2L$.

Case 4) $0 \le i \le M-1$, $M \le j \le 2M-1$ (or $M \le i \le 2M-1$, $0 \le j \le M-1$), and τ is odd.

It is easy to see that $\omega^{-q/2} = -1$. Therefore, $R_{i,j}(\tau)$ can be rewritten as

$$\begin{aligned} R_{i,j}(\tau) &= \sum_{t=0}^{N-1} \omega^{v_i(t) - v_{j-M}(t+\frac{L+1}{2}+\frac{\tau-1}{2}) - \frac{q}{2}} + \sum_{t=0}^{N-1} \omega^{v_i(t+\frac{L+1}{2}) - v_{j-M}(t+\frac{\tau+1}{2})} \\ &= -\sum_{t=0}^{N-1} \omega^{v_i(t) - v_{j-M}(t+\frac{L+1}{2}+\frac{\tau-1}{2})} + \sum_{t=0}^{N-1} \omega^{v_i(t+\frac{L+1}{2}) - v_{j-M}(t+\frac{\tau+1}{2})}. \end{aligned}$$

Similarly to Case 2), $|R_{i,j}(\tau)| \le 2\epsilon$ within the range $-L < \tau < L$.

Case 5) $M \le i, j \le 2M-1$ and τ is even.

In this case, $R_{i,j}(\tau)$ can be rewritten as

$$R_{i,j}(\tau) = \sum_{t=0}^{N-1} \omega^{v_{i-M}(t) - v_{j-M}(t+\frac{\tau}{2})} + \sum_{t=0}^{N-1} \omega^{v_{i-M}(t+\frac{L+1}{2}) - v_{j-M}(t+\frac{L+1}{2}+\frac{\tau}{2})}.$$

Similarly to Case 1), $|R_{i,j}(\tau)| \le 2\epsilon$ within the range $-2L < \tau < 2L$.

Case 6) $M \le i, j \le 2M-1$ and τ is odd.

In this case, $R_{i,j}(\tau)$ can be rewritten as

$$R_{i,j}(\tau) = \sum_{t=0}^{N-1} \omega^{v_{i-M}(t) - v_{j-M}(t+\frac{L+1}{2}+\frac{\tau-1}{2})} + \sum_{t=0}^{N-1} \omega^{v_{i-M}(t+\frac{L+1}{2}) - v_{j-M}(t+\frac{\tau+1}{2})}.$$

Similarly to Case 2), $|R_{i,j}(\tau)| \le 2\epsilon$ within the range $-L < \tau < L$.

From the above 6 cases, it is clear that $\mathcal{T}_1$ is an LCZ sequence set with parameters $(2N, 2M, L, 2\epsilon)$, if L is odd. □

To the best of our knowledge, the only known optimal q-ary LCZ sequence sets with parameters $(N, M, L, 1)$ and even q are due to [7] for $q = 2$ and [6] for $q = 4$.

When we apply Construction 3 to the above optimal binary and quaternary LCZ sequence sets, it turns out that resulting LCZ sequence sets with parameters $(2N, 2M, L, 2)$ become optimal.

Example 12: Let $n = 4$, $e = m = 2$, and $T = (2^n - 1)/(2^e - 1) = 5$. Let α be a primitive element in the finite field F_{2^4} with 16 elements. Let $\mathcal{L}_1$ be the quaternary LCZ sequence set with parameters $(15, 3, 5, 1)$ given as

$$\mathcal{L}_1 = \{m_i(t) \mid 0 \le i \le 2,\ 0 \le t < 15\}$$
$$m_i(t) = \begin{cases} 2\mathrm{tr}_1^4(\alpha^t) & \text{for } i = 0 \\ \mathrm{tr}_1^4(\alpha^t) \boxplus 2\mathrm{tr}_1^4(\alpha^{t+5i}) & \text{otherwise} \end{cases}$$

where $\boxplus$ denotes addition modulo 4 and $\mathrm{tr}_1^4(\cdot)$ is a trace function from F_{2^4} to F_2.

Then the following $\mathcal{T}_1$ is the optimal quaternary LCZ sequence set with parameters $(30, 6, 5, 2)$

$$\mathcal{T}_1 = \{s_i(t) \mid 0 \le i \le 5,\ 0 \le t < 30\}$$

where $s_i(t)$ is given as

$$\begin{aligned} s_0(t) &= 020000220200222002220222202020 \\ s_1(t) &= 012220132302333021310311103212 \\ s_2(t) &= 032220312102111023130133301232 \\ s_3(t) &= 222020022222002002202222020000000 \\ s_4(t) &= 210200330322131001112331301232 \\ s_5(t) &= 230200110122313003332113103212. \end{aligned}$$
□

The following corollary gives us the condition for the parameters N, M, and L of the component LCZ sequence set so that the resulting $(2N, 2M, L, 2)$ LCZ sequence set becomes optimal.

Corollary 13: Assume that the component sequence set is an optimal $(N, M, L, 1)$ LCZ sequence set with an odd integer L. If the following relation holds among N, M, and L as

$$N-(L-2) < ML \leq N+1$$

then the new q-ary LCZ sequence set constructed by Construction 3 is also optimal in the sense that larger set cannot exist for given N, L, and $\epsilon = 2$.

Proof: Plugging $\epsilon = 1$ into the bound in Theorem 5, we have

$$M \leq \frac{N+1}{L}.$$

Therefore, the optimality in terms of the set size of the component LCZ sequence set implies

$$M \leq \frac{N+1}{L} < M+1$$

which in turn becomes

$$N-(L-1) < ML \leq N+1. \tag{23}$$

Replacing the parameters by $(2N, 2M, L, 2)$ in the bound in Theorem 5 gives us

$$ML \leq \frac{N^2-1}{N-2} = N+2+\frac{3}{N-2}.$$

Therefore, in order for the new LCZ sequence set to be optimal, the range of ML should be

$$ML \leq N+2+\frac{3}{N-2} < (M+1)L.$$

From the above equation, we have

$$N-(L-2) < ML < N+3. \tag{24}$$

From (23) and (24), it is clear that the range of ML that makes both the component sequence set and the new sequence set optimal is

$$N-(L-2) < ML \leq N+1. \qquad \square$$

Especially for the earlier mentioned optimal $(N, M, L, 1)$ LCZ sequence sets in [7] and [6], we have another extension method to apply. This second extension method, later called Construction 4 also yields the optimal extended $(2N, 2M, L, 2)$ binary LCZ sequence set.

Let $\mathcal{L}_2$ be a set of the binary LCZ sequences with parameters $(N, M, L, 1)$, where $N \equiv 3 \bmod 4$ and $L|N$. Assume that the correlation functions $R_{i,j}(\tau)$ between any two sequences $v_i(t)$ and $v_j(t)$ in $\mathcal{L}_2$ take the value -1 all the time except for $\tau \equiv 0 \bmod L$ ($\tau \neq 0$). Also assume that all the sequences in $\mathcal{L}_2$ are balanced.

Let D_i be the characteristic set of the LCZ sequence $v_i(t)$ in $\mathcal{L}_2$. Define two subsets $E_i^{(1)}$ and $E_i^{(2)}$ of $Z_2 \otimes Z_N$ as

$$\begin{aligned} E_i^{(1)} &= \{0\} \otimes D_i \cup \{1\} \otimes (D_i + L) \\ E_i^{(2)} &= \{0\} \otimes D_i \cup \{1\} \otimes (\overline{D}_i + L). \end{aligned}$$

Let $\mathcal{V}_k$, $k = 1, 2$, be the sets of all the characteristic sequences $s_i^{(k)}(t)$ of the sets $E_i^{(k)}$, $i = 0, 1, \ldots, M-1$, respectively.

Construction 4: The new binary LCZ sequence set is defined as

$$\begin{aligned} \mathcal{T}_2 &= \mathcal{V}_1 \cup \mathcal{V}_2 \\ &= \{s_i^{(k)}(t) \mid 0 \leq i \leq M-1,\ 0 \leq t < 2N,\ k = 1, 2\}. \qquad \square \end{aligned}$$

Theorem 14: $\mathcal{T}_2 = \mathcal{V}_1 \cup \mathcal{V}_2$ is a binary LCZ sequence set with parameters $(2N, 2M, L, 2)$.

Proof: Let $\tau = (\tau_1, \tau_2) \in Z_2 \otimes Z_N$. We have to consider three cases. First, consider the case that $s_i^{(1)}(t)$ and $s_j^{(1)}(t)$ are in $\mathcal{V}_1$. Similarly to (8), we have the first equation shown at the bottom of the page. From the balance property and correlation value -1 of the component LCZ sequence set for $\tau \not\equiv 0 \bmod L$, we have

$$\begin{aligned} |D_i \cap (D_j + \tau_2)| = |D_i \cap (D_j + L + \tau_2)| &= \frac{(N+1)}{4} \\ |(D_i + L) \cap (D_j + L + \tau_2)| = |(D_i + L) \cap (D_j + \tau_2)| &= \frac{(N+1)}{4} \end{aligned}$$

for $\tau_2 \not\equiv 0 \bmod L$. Thus, we have

$$\begin{aligned} d_{i,j}(\tau) &= \frac{(N+1)}{2} \\ R_{i,j}(\tau) &= -2 \end{aligned}$$

at $\tau \not\equiv 0 \bmod L$. When $\tau = 0$, we have

$$d_{i,j}(\tau) = |D_i \cap D_j| + |(D_i + L) \cap (D_j + L)|.$$

Similarly, it is clear that

$$|D_i \cap D_j| = |(D_i + L) \cap (\overline{D}_j + L)| = \frac{(N+1)}{4}$$

for $i \neq j$. Therefore, we also have

$$\begin{aligned} d_{i,j}(0) &= \frac{(N+1)}{2} \\ R_{i,j}(0) &= -2. \end{aligned}$$

Second, for the case that $s_i^{(1)}(t)$ is in $\mathcal{V}_1$ and $s_j^{(2)}(t)$ in $\mathcal{V}_2$, we have the first equation shown at the top of the following page. Similarly to the first case, we have

$$\begin{aligned} |D_i \cap (D_j + \tau_2)| = |D_i \cap (\overline{D}_j + L + \tau_2)| &= \frac{(N+1)}{4} \\ |(D_i + L) \cap (\overline{D}_j + L + \tau_2)| = |(D_i + L) \cap (D_j + \tau_2)| &= \frac{(N+1)}{4} \end{aligned}$$

$$d_{i,j}(\tau) = |E_i^{(1)} \cap (E_j^{(1)} + \tau)| = \begin{cases} |D_i \cap (D_j + \tau_2)| + |(D_i + L) \cap (D_j + L + \tau_2)|, & \text{for } \tau_1 = 0 \\ |D_i \cap (D_j + L + \tau_2)| + |(D_i + L) \cap (D_j + \tau_2)|, & \text{for } \tau_1 = 1. \end{cases}$$

$$d_{i,j}(\tau) = |E_i^{(1)} \cap (E_j^{(2)} + \tau)| = \begin{cases} |D_i \cap (D_j + \tau_2)| + |(D_i + L) \cap (\overline{D}_j + L + \tau_2)|, & \text{for } \tau_1 = 0 \\ |D_i \cap (\overline{D}_j + L + \tau_2)| + |(D_i + L) \cap (D_j + \tau_2)|, & \text{for } \tau_1 = 1. \end{cases}$$

$$d_{i,j}(\tau) = |E_i^{(2)} \cap (E_j^{(2)} + \tau)| = \begin{cases} |D_i \cap (D_j + \tau_2)| + |(\overline{D}_i + L) \cap (\overline{D}_j + L + \tau_2)|, & \text{for } \tau_1 = 0 \\ |D_i \cap (\overline{D}_j + L + \tau_2)| + |(\overline{D}_i + L) \cap (D_j + \tau_2)|, & \text{for } \tau_1 = 1. \end{cases}$$

for $\tau_2 \not\equiv 0 \bmod L$, which yields

$$d_{i,j}(\tau) = \frac{(N+1)}{2}$$
$$R_{i,j}(\tau) = 0.$$

For $\tau = 0$, it is easy to derive $R_{i,j}(0) = 0$.

Finally, for the case that $s_i^{(2)}(t)$ and $s_j^{(2)}(t)$ are in $\mathcal{V}_2$, we have the second equation shown at the top of the page. Similarly to the previous cases, it is easy to compute $R_{i,j}(\tau) = \pm 2$ except for the in-phase autocorrelation for $-L < \tau < L$. □

We can easily obtain the following corollary.

Corollary 15: $\mathcal{T}_2$ is optimal if and only if the component binary LCZ sequence set $\mathcal{L}_2$ is optimal.

Proof: From Corollary 13 and the fact that L divides N, it is clear that the component LCZ sequence set and its extended LCZ sequence set is optimal if and only if $N = LM$. Therefore, $\mathcal{T}_2$ is optimal if and only if the component binary LCZ sequence set $\mathcal{L}_2$ is optimal. □

Examples of component LCZ sequence sets to which Construction 4 can be applied are found in [7] and [3]. In the next example, we will construct an optimal $(510, 30, 17, 2)$ binary LCZ sequence set from the $(255, 15, 17, 1)$ component optimal binary LCZ sequence set given in [7].

Example 16: Let $m(t)$ be the m-sequence of period 15 given as

$$m(t) = 000100110101111.$$

From the result of [7], we can construct the binary LCZ sequence set of period 255 using the binary column sequence set given as

$$\mathcal{S} = \{b_i(t) \mid 0 \le i \le 14,\ 0 \le t < 15\}$$

where $b_i(t)$ is given as

$$b_i(t) = \begin{cases} m(-t + i + 14), & \text{for } 0 \le t \le 7 \\ m(t + i + 7), & \text{for } 8 \le t < 15. \end{cases}$$

Let α be a primitive element in F_{2^8}. Let $c_i(\beta^t) = b_i(t)$, where β is a primitive element in F_{2^4}. Using the column sequence set $\mathcal{S}$, we can construct the component optimal binary LCZ set $\mathcal{L}_2$ with parameters $(255, 15, 17, 1)$ as

$$\mathcal{L}_2 = \{l_i(t) \mid 0 \le i \le 14,\ 0 \le t < 255\}$$

where $l_i(t)$ is given as

$$l_i(t) = c_i(\mathrm{tr}_4^8(\alpha^t)).$$

Using the component LCZ sequence set $\mathcal{L}_2$, we can construct the optimal binary LCZ sequence set $\mathcal{T}_2$ with parameters $(510, 30, 17, 2)$ as

$$\mathcal{T}_2 = \{s_i^{(k)}(t) \mid 0 \le i \le 14,\ 0 \le t < 510,\ k = 1, 2\}$$
$$s_i^{(k)}(t) = \begin{cases} l_i(t), & \text{for } 0 \le t < 255,\ t \text{ even} \\ l_i(t - 255), & \text{for } 255 \le t < 510,\ t \text{ even} \\ l_i(t + 17), & \text{for } k = 1,\ 0 \le t < 255,\ t \text{ odd} \\ l_i(t + 17) \oplus 1, & \text{for } k = 2,\ 0 \le t < 255,\ t \text{ odd} \\ l_i(t - 238), & \text{for } k = 1,\ 255 \le t < 510,\ t \text{ odd} \\ l_i(t - 238) \oplus 1, & \text{for } k = 2,\ 255 \le t < 510,\ t \text{ odd} \end{cases}$$

where $\oplus$ is addition modulo 2 and argument in $l_i(\cdot)$ is computed modulo 255. □

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