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Butson Hadamard matrices with partially cyclic core

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Abstract In this paper, we introduce a class of generalized Hadamard matrices, called a Butson Hadamard matrix with partially cyclic core. Then a new construction method for Butson Hadamard matrices with partially cyclic core is proposed. The proposed matrices are constructed from the optimal balanced low-correlation zone(LCZ) sequence set which has correlation value -1 within LCZ.

Keywords Butson Hadamard matrices with partially cyclic core · Generalized Hadamard matrices · Hadamard matrices · Low-correlation zone(LCZ) sequence sets · Sequences

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1 Introduction

A Hadamard matrix H of order u is a $u \times u$ matrix of $+1$'s and -1's satisfying

$$HH^t = uI_u,$$

where H^t denotes the transpose of H and I_u is the identity matrix of order u [1]. In 1867, Sylvester proposed a recurrent method for construction of Hadamard matrices of order 2^k. Paley conjectured in 1933 that there exists a Hadamard matrix of any order divisible by 4, which

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has not been proved or disproved yet [2,3]. There are many different types of constructions for Hadamard matrices of order $4n$, such as Sylvester construction, Paley construction, Turyn construction, and Williamson construction [4]. The definition of Hadamard matrix implies that any two distinct rows of H are orthogonal. For this reason, Hadamard matrices have been studied for the applications in many areas such as wireless communication systems, coding theory, signal design, and image processing [2].

A Butson Hadamard matrix, $\mathcal{H}(q, u)$ of order u is a $u \times u$ matrix over the set of complex qth roots of unity satisfying

$$\mathcal{H}(q, u)\mathcal{H}^{\dagger}(q, u) = uI_u$$

where † denotes the conjugate transpose [3]. For brevity, we use the notation $\mathcal{H}$ interchangeably with $\mathcal{H}(q, u)$ if specifying q and u is unnecessary. When we mention the generalized Hadamard matrix in this paper, it also comprises Hadamard matrix.

In the reverse link of code division multiple access(CDMA) systems, synchronization within a few chips can be maintained due to the relatively small transmission delay, in which spreading sequences with good correlation property around origin are needed. Such a system is called a quasi-synchronous(QS) CDMA system [5]. Recently, there have been many research results on the low-correlation zone(LCZ) sequences, which can be used as a spreading sequence in the QS CDMA systems [5–8]. We find it interesting that a class of Butson Hadamard matrices can be constructed from the optimal balanced LCZ sequence sets with nontrivial correlation value -1.

In this paper, we define Butson Hadamard matrices with partially cyclic core. Then a new construction method for Butson Hadamard matrices with partially cyclic core is proposed. The proposed matrices are constructed from the optimal balanced LCZ sequence set which has correlation value -1 within LCZ.

2 Preliminaries

For an integer q, let N be a positive integer such that $N \equiv -1 \bmod q$ and $s(t)$ a q-ary sequence of period N. Then $s(t)$ is said to be *balanced* if the number of occurrences of the symbol 0 in one period is exactly one less than that of any other nonzero symbols in Z_q. It is clear that the balanced q-ary sequence $s(t)$ of period N has the property

$$\sum_{t=0}^{N-1} \omega^{s(t)} = -1,$$

where ω is a primitive complex qth root of unity.

Given a q-ary sequence $s(t)$ of period N, the autocorrelation $R_s(\tau)$ of the sequence at shift τ is defined by

$$R_s(\tau) = \sum_{t=0}^{N-1} \omega^{s(t)-s(t+\tau)}.$$

A q-ary sequence $s(t)$ of period N is said to have ideal autocorrelation property [12] if

$$R_s(\tau) = \begin{cases} N, & \text{if } \tau = 0 \\ -1, & \text{otherwise.} \end{cases}$$

Two q-ary sequences $u(t)$ and $v(t)$ of the same period N are said to be cyclically equivalent if there exists some integer $\tau(\neq 0 \mod N)$ such that $u(t) = v(t+\tau)$ for all t. Two sequences which are not cyclically equivalent are said to be cyclically inequivalent. Given two q-ary sequences $u(t)$ and $v(t)$ of the same period N, the crosscorrelation $R_{u,v}(\tau)$ between $u(t)$ and $v(t)$ is defined by

$$R_{u,v}(\tau) = \sum_{t=0}^{N-1} \omega^{u(t)-v(t+\tau)}.$$

The sequence $\omega^{s(t)}$ can be considered as the complex counterpart of $s(t)$. Throughout the rest of this paper, when we mention a sequence with some correlation property, we interchangeably imply $s(t)$ or $\omega^{s(t)}$ if no confusion is caused by the context.

Klapper [9] introduced the d-form function. A d-form function $H(x)$ from F_{p^n} into F_{p^m} is defined as a function satisfying for any $y \in F_{p^m}$ and $x \in F_{p^n}$

$$H(yx) = y^d H(x).$$

A Butson Hadamard matrix with cyclic core is defined as:

Definition 1 (Butson Hadamard matrix with a cyclic core) Let $s(t)$ be a q-ary sequence of period N with ideal autocorrelation property. Let $\mathcal{H}_C$ be an $(N+1) \times (N+1)$ matrix given by

$$\mathcal{H}_C(q, N+1) = (h_{ij}).$$

Then the matrix $\mathcal{H}_C$ is called a Butson Hadamard matrix with a cyclic core $s(t)$ if

$$h_{ij} = \begin{cases} 1, & \text{if } i = 0 \text{ or } j = 0 \\ \omega^{s(i+j-2)}, & \text{otherwise.} \end{cases}$$

□

From the Definition 1, it is clear that an $\mathcal{H}_C(q, N+1)$ is completely characterized by a q-ary sequence $s(t)$ of period N with ideal autocorrelation, and vice versa. And in this sense, we may call this sequence the cyclic core associated with the Butson Hadamard matrix.

Now, let us broaden this idea of association, i.e., a Butson Hadamard matrix associated with a set of sequences instead of a single sequence. In this context, we can define a Butson Hadamard matrix with partially cyclic core as follows.

Definition 2 (Butson Hadamard matrix with partially cyclic core) Let $\mathcal{S} = \{s_i(t) \mid 0 \leq i \leq M-1\}$ be a set of M cyclically inequivalent q-ary sequences of period N, such that $M|N$. Let $N = eM$. Let $\mathcal{H}_{PC}$ be an $(N+1) \times (N+1)$ matrix defined by

$$\mathcal{H}_{PC}(q, N+1) = (c_{ij}),$$

where c_{ij} is given as

$$c_{ij} = \begin{cases} 1, & \text{if } i = 0 \text{ or } j = 0 \\ \omega^{s_{i_q}(j+i_r-1)}, & \text{otherwise} \end{cases}$$

and i_q and i_r are the quotient and the remainder of $(i-1)$ divided by e, respectively, i.e., $i - 1 = i_q e + i_r$. Then the matrix $\mathcal{H}_{PC}$ is called a *Butson Hadamard matrix with partially cyclic core* $\mathcal{S}$ if $\mathcal{H}_{PC}\mathcal{H}_{PC}^{\dagger} = (N+1)I_{N+1}$. □

3 Constructions for Butson Hadamard matrices with partially cyclic core

In this section, we propose a new construction method for Butson Hadamard matrices associated with an optimal balanced LCZ sequence set that has the correlation value -1 within LCZ.

Definition 3 (LCZ sequence set [5]) Let $\mathcal{S}$ be a set of M sequences of period N. If the magnitude of the out-of-phase($\tau \neq 0$) autocorrelation as well as the crosscorrelation between any two sequences in $\mathcal{S}$ takes values not exceeding a given positive value ϵ for every offset τ within the range $-L < \tau < L$, then $\mathcal{S}$ is called an LCZ sequence set with parameters (N, M, L, ϵ) and L is called the LCZ. □

Tang et al. [6] derived the lower bound on the size of an LCZ sequence set using the Welch bound [13].

Theorem 1 (Tang et al. [6]) *Let $\mathcal{S}$ be an LCZ sequence set with parameters (N, M, L, ϵ). Then,*

$$M \leq \left\lfloor \frac{N^2 - \epsilon^2}{L(N - \epsilon^2)} \right\rfloor$$

where $\lfloor x \rfloor$ denotes the greatest integer not exceeding x. □

An LCZ sequence set achieving the equality in the above bound is called an *optimal LCZ sequence set*. And if all the sequences in the LCZ sequence set are balanced, we call it a *balanced LCZ sequence set*. Associated with the optimal balanced LCZ sequence set that has correlation value -1 within the LCZ, we can construct a Butson Hadamard matrix with partially cyclic core as in the following theorem.

Theorem 2 *Let L, M, and N be integers such that $N = ML$. Let $\mathcal{S} = \{s_i(t) | 0 \leq i \leq M - 1, 0 \leq t \leq N - 1\}$ be an optimal balanced LCZ sequence set with parameters $(N, M, L, 1)$. Suppose that the correlation value between any two sequences in $\mathcal{S}$ within the LCZ is -1. Then we can construct an $(N + 1) \times (N + 1)$ Butson Hadamard matrix with partially cyclic core*

$$\mathcal{H}_{PC}(q, N + 1) = (h_{jk}),$$

where h_{jk} is given as

$$h_{jk} = \begin{cases} 1, & \text{if } j = 0 \text{ or } k = 0 \\ w^{s_{\lfloor (j-1)/L \rfloor}(k-1+j_L)}, & \text{otherwise} \end{cases}$$

and $j_L = (j - 1)$ mod L.

Proof Let H^s_{PC} be the $N \times N$ submatrix of H_{PC} obtained by deleting the first row and the first column of H_{PC}.What we are going to show is that each row in $\mathcal{H}_{PC}$ is orthogonal to every other row in $\mathcal{H}_{PC}$ and each member in $\mathcal{S}$ appears exactly the same number of times as some row of $\mathcal{H}^s_{PC}$ in the form of its cyclic shifts (including zero shift).

Since rows in $\mathcal{H}^s_{PC}$ are cyclic shifts of the sequences in $\mathcal{S}$, it is clear that all rows in $\mathcal{H}^s_{PC}$ are balanced. From the definition of h_{jk}, it is clear that all sequences in $\mathcal{S}$ have the same number of occurrences as rows in $\mathcal{H}^s_{PC}$.

Let v_i be the ith row of $\mathcal{H}_{PC}$, $0 \leq i \leq N$. We have to show that $v_i v_k^\dagger = 0$ for all $i \neq k$. Since v_0 is an all one sequence and each v_i for $1 \leq i \leq N$ comes from a balanced sequence, it is clear that $v_0 v_i^\dagger = 0$ for all $1 \leq i \leq N$. From the structure of $\mathcal{H}_{PC}$, it is clear that

the rows v_{1+lL} through v_{L+lL}, $0 \le l \le M-1$, are the cyclic shifts of $\omega^{s_l(t)}$. And for all $1 \le i < k \le N$, we can rewrite $v_i v_k^{\dagger}$ as follows

$$v_i v_k^{\dagger} = 1 + \sum_{t=0}^{N-1} w^{s_{\lfloor (i-1)/L \rfloor}(t+\tau_i) - s_{\lfloor (k-1)/L \rfloor}(t+\tau_k)},$$

where $\tau_i = i - 1 - \lfloor (i-1)/L \rfloor L$ and $\tau_k = k - 1 - \lfloor (k-1)/L \rfloor L$.

From the property of LCZ sequence set with correlation value -1 within the low-correlation zone, it is clear that

$$\sum_{t=0}^{N-1} w^{s_{\lfloor (i-1)/L \rfloor}(t+\tau_i) - s_{\lfloor (k-1)/L \rfloor}(t+\tau_k)} = -1.$$

□

In [10], an optimal balanced LCZ sequence set with parameters $(p^n-1, p^m-1, (p^n-1)/(p^m-1), 1)$ was constructed using a 1-form function from F_{p^n} to F_{p^m}. Applying Theorem 2 to this construction, we have the following example.

Example 1 Let m and n be integers such that $m|n$. Let p be a prime. Let α and β be primitive elements in F_{p^n} and F_{p^m}, respectively, satisfying $\beta = \alpha^{\frac{p^n-1}{p^m-1}}$. Let $v(\cdot)$ be a 1-form function from F_{p^m} to F_p and $f(\cdot)$ a 1-form function from F_{p^n} to F_{p^m}. Let $\mathcal{S}_p$ be the optimal p^2-ary LCZ sequence set with parameters $(p^n-1, p^m-1, (p^n-1)/(p^m-1), 1)$ given by

$$\mathcal{S}_p = \{s_i(t) \mid 0 \le i \le p^m - 2\},$$

where $s_i(t)$ is defined by

$$s_i(t) = \begin{cases} pf([v(\beta^i \alpha^t)]^r), & \text{if } \beta^i \in F_p \\ f([v(\alpha^t)]^r) + pf([v(\beta^i \alpha^t)]^r), & \text{otherwise.} \end{cases}$$

Then we can construct a $p^n \times p^n$ Butson Hadamard matrix with partially cyclic core

$$\mathcal{H}_{PC}(p^2, p^n) = (h_{jk}),$$

where h_{jk} is defined by

$$h_{jk} = \begin{cases} 1, & \text{if } j = 0 \text{ or } k = 0 \\ w^{s_{\lfloor (j-1)/L \rfloor}(k-1+j_L)}, & \text{otherwise} \end{cases}$$

and $L = (p^n-1)/(p^m-1)$ and $j_L = (j-1) \bmod L$. □

As one of the simplest form of Example 1, a 16×16 quaternary Butson Hadamard matrix with partially cyclic core is shown below.

Example 2 Let α be a primitive element in F_{2^4}. Let $\mathcal{S}_b$ be the optimal quaternary LCZ sequence set [8] with parameters $(15, 3, 5, 1)$ defined by

$$\mathcal{S}_b = \{s_i(t) \mid 0 \le i \le 2\},$$

where $s_i(t)$ is defined by

$$s_i(t) = \begin{cases} 2\mathrm{tr}_1^4(\alpha^t), & \text{if } i = 0 \\ \mathrm{tr}_1^4(\alpha^t) + 2\mathrm{tr}_1^4(\alpha^{t+5i}), & \text{if } i = 1 \text{ or } 2 \end{cases}$$

and $tr_1^4(\cdot)$ denotes trace function from F_{2^4} to F_2 [1].

Then the following matrix $\mathcal{H}_{PC}$, where the entry $(\sqrt{-1})^c$ is denoted by c, is the quaternary Butson Hadamard matrix with partially cyclic core $\mathcal{S}_b$.

$$\mathcal{H}_{PC}(4,16) = \left[\begin{array}{cccccccccccccccc}
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&2&0&0&2&2&0&2&0&2&2&2&2\\
0&0&0&2&0&0&2&2&0&2&0&2&2&2&2&0\\
0&0&2&0&0&2&2&0&2&0&2&2&2&2&0&0\\
0&2&0&0&2&2&0&2&0&2&2&2&2&0&0&0\\
0&0&0&2&2&0&2&0&2&2&2&2&0&0&0&2\\
0&0&2&2&1&2&0&3&3&2&3&0&1&1&3&1\\
0&2&2&1&2&0&3&3&2&3&0&1&1&3&1&0\\
0&2&1&2&0&3&3&2&3&0&1&1&3&1&0&2\\
0&1&2&0&3&3&2&3&0&1&1&3&1&0&2&2\\
0&2&0&3&3&2&3&0&1&1&3&1&0&2&2&1\\
0&0&2&2&3&2&0&1&1&2&1&0&3&3&1&3\\
0&2&2&3&2&0&1&1&2&1&0&3&3&1&3&0\\
0&2&3&2&0&1&1&2&1&0&3&3&1&3&0&2\\
0&3&2&0&1&1&2&1&0&3&3&1&3&0&2&2\\
0&2&0&1&1&2&1&0&3&3&1&3&0&2&2&3
\end{array}\right].$$

□

Note that rows r_1 through r_5 come from $s_0(t)$, r_6 through r_{10} from $s_1(t)$, and r_{11} through r_{15} from $s_2(t)$.

Jang, No, Chung, and Tang proposed a construction method for optimal p-ary LCZ sequence set [11]. Using their result in [11], we are able to apply Theorem 2 to give the following example.

Example 3 Let m and n be integers such that $m|n$. Let p be a prime and $f(t)$ a p-ary sequence of period $p^m - 1$ with ideal autocorrelation. Let $\{f_i(t) \mid 0 \le i \le p^m - 2\}$ be a set of cyclic shifts of $f(t)$, such that $f_i(t) = f(t+i),\ t = 0, 1, 2, \cdots, p^m - 2$. From the result of [11], we can construct the so called column sequence set $\mathcal{V}$ for p-ary LCZ sequence set of period $p^n - 1$ as

$$\mathcal{V} = \{v_i(t) \mid 0 \le i \le p^m - 2\},$$

where $v_i(t)$ is defined by

$$v_i(t) = \begin{cases} f_i(p^m - 2 - 1 - t), & 0 \le t \le \frac{p^m-1}{2} \\ f_i(t + (p^m - 1)/2), & \frac{p^m-1}{2} + 1 \le t \le p^m - 2. \end{cases}$$

Let α be a primitive element in F_{p^n}. Let $c_i(\beta^t) = v_i(t)$, where β is a primitive element in F_{p^m}. Using the column sequence set $\mathcal{V}$, we can construct the optimal p-ary LCZ set $\mathcal{S}_p$ with parameters $(p^n - 1, p^m - 1, (p^n - 1)/(p^m - 1), 1)$ by

$$\mathcal{S}_p = \{s_i(t) \mid 0 \le i \le p^m - 2 \text{ and } 0 \le t \le p^n - 2\},$$

where $s_i(t)$ is defined by

$$s_i(t) = c_i(\mathrm{tr}_m^n(\alpha^t)).$$

Applying Theorem 2 to the LCZ sequence set $\mathcal{S}_p$, we can construct the Butson Hadamard matrix with partially cyclic core

$$\mathcal{H}_{PC}(p, p^n) = (h_{jk}),$$

where h_{jk} is defined by

$$h_{jk} = \begin{cases} 1, & \text{if } j = 0 \text{ or } k = 0 \\ w^{s_{\lfloor (j-1)/L \rfloor}(k-1+j_L)}, & \text{otherwise} \end{cases}$$

and $L = (p^n - 1)/(p^m - 1)$ and $j_L = (j - 1) \bmod L$. □

Here is an example of a 256×256 Butson Hadamard matrix with partially cyclic core with $p = 2$, $m = 4$, and $n = 8$ in Example 3.

Example 4 Let $f(t)$ be the binary m-sequence of period 15 with ideal autocorrelation given by

$$f(t) = 0, 0, 0, 1, 0, 0, 1, 1, 0, 1, 0, 1, 1, 1, 1.$$

From the result of [11], we can construct the column sequence set $\mathcal{V}$ for a binary LCZ sequence set of period 255 by

$$\mathcal{V} = \{v_i(t) \mid 0 \le i \le 14\},$$

where $v_i(t)$ is given below

$$\begin{aligned}
v_0(t) &= 1\ 1\ 1\ 1\ 0\ 1\ 0\ 0\ 0\ 0\ 1\ 0\ 0\ 1\ 1 \\
v_1(t) &= 0\ 1\ 1\ 1\ 1\ 0\ 1\ 0\ 0\ 1\ 0\ 0\ 1\ 1\ 0 \\
v_2(t) &= 0\ 0\ 1\ 1\ 1\ 1\ 0\ 0\ 1\ 0\ 0\ 1\ 1\ 0\ 1 \\
v_3(t) &= 0\ 0\ 0\ 1\ 1\ 1\ 1\ 1\ 0\ 0\ 1\ 1\ 0\ 1\ 0 \\
v_4(t) &= 1\ 0\ 0\ 0\ 1\ 1\ 1\ 0\ 0\ 1\ 1\ 0\ 1\ 0\ 1 \\
v_5(t) &= 0\ 1\ 0\ 0\ 0\ 1\ 1\ 0\ 1\ 1\ 0\ 1\ 0\ 1\ 1 \\
v_6(t) &= 0\ 0\ 1\ 0\ 0\ 0\ 1\ 1\ 1\ 0\ 1\ 0\ 1\ 1\ 1 \\
v_7(t) &= 1\ 0\ 0\ 1\ 0\ 0\ 0\ 1\ 0\ 1\ 0\ 1\ 1\ 1\ 1 \\
v_8(t) &= 1\ 1\ 0\ 0\ 1\ 0\ 0\ 0\ 1\ 0\ 1\ 1\ 1\ 1\ 0 \\
v_9(t) &= 0\ 1\ 1\ 0\ 0\ 1\ 0\ 1\ 0\ 1\ 1\ 1\ 1\ 0\ 0 \\
v_{10}(t) &= 1\ 0\ 1\ 1\ 0\ 0\ 1\ 0\ 1\ 1\ 1\ 1\ 0\ 0\ 0 \\
v_{11}(t) &= 0\ 1\ 0\ 1\ 1\ 0\ 0\ 1\ 1\ 1\ 1\ 0\ 0\ 0\ 1 \\
v_{12}(t) &= 1\ 0\ 1\ 0\ 1\ 1\ 0\ 1\ 1\ 1\ 0\ 0\ 0\ 1\ 0 \\
v_{13}(t) &= 1\ 1\ 0\ 1\ 0\ 1\ 1\ 1\ 1\ 0\ 0\ 0\ 1\ 0\ 0 \\
v_{14}(t) &= 1\ 1\ 1\ 0\ 1\ 0\ 1\ 1\ 0\ 0\ 0\ 1\ 0\ 0\ 1\ .
\end{aligned}$$

Let α be a primitive element in F_{2^8}. Let $c_i(\beta^t) = v_i(t)$, where β is a primitive element in F_{2^4}. Using the column sequence set $\mathcal{V}$, we can construct the optimal binary LCZ set $\mathcal{S}_2$ with parameters (255, 15, 17, 1) as

$$\mathcal{S}_2 = \{s_i(t) \mid 0 \le i \le 14 \quad \text{and} \quad 0 \le t \le 254\},$$

where $s_i(t)$ is given by

$$s_i(t) = c_i(\mathrm{tr}_4^8(\alpha^t)).$$

The hexadecimal representations of $s_i(t)$, $0 \le i \le 14$, $0 \le t \le 254$, are given as

$$s_0(t) = 28cb\ ddfc\ a1b3\ 6551\ 2827\ 8035\ b85c\ f008\ 692e\ 364a\ bb05\ 61e7\ 275b\ d46d\ de1e\ e727$$
$$s_1(t) = 3b70\ d763\ ee72\ 501f\ 32eb\ f4ba\ 600a\ 0654\ 7495\ 760b\ dacb\ 0f35\ 88c9\ 3f30\ ac7c\ e42b$$
$$s_2(t) = 3f0e\ dca0\ 6abc\ 759f\ e95c\ d015\ ccb8\ 3926\ 8413\ b196\ 1f6f\ 5d10\ deb2\ c8f4\ 8a50\ 4ece$$
$$s_3(t) = 1102\ 9d99\ 6f5d\ b7b8\ ba8f\ 9d66\ 7a4f\ 2c27\ 132a\ 0add\ b821\ bd65\ 97a8\ 47c0\ 35b8\ 5c38$$
$$s_4(t) = 13bb\ 0a9f\ 4fc1\ 354e\ 1acc\ 748f\ d856\ f65c\ 1dbb\ 4041\ 61ce\ 6ed2\ af92\ eb5d\ 7262\ 030c$$
$$s_5(t) = 047e\ 0bc3\ 84ce\ 2580\ dbb7\ 24af\ acb2\ 3f72\ f086\ c79d\ c5a4\ 5225\ 567b\ f7c4\ 262c\ aae5$$
$$s_6(t) = 2e0c\ 4139\ 05e1\ c227\ 53d3\ 4d73\ b6f7\ 1501\ 9739\ bb4b\ a74e\ e075\ 491a\ 8f34\ bfe8\ 12f6$$
$$s_7(t) = 02b9\ 9706\ 209c\ 82f6\ a043\ e9e9\ a219\ da7b\ 0e91\ 4a9c\ d9ef\ d3b7\ 383a\ ac9d\ 47da\ 5f34$$
$$s_8(t) = 17c5\ 015c\ cb0f\ 10ce\ c17b\ 5020\ 74e4\ c92e\ ed3d\ 87dc\ a46a\ 3cf7\ f9e9\ 1c99\ 544e\ a9e9$$
$$s_9(t) = 2a72\ 4afa\ 812f\ e7a7\ 8864\ 69dc\ 1a45\ 2a73\ 67bf\ 7cd6\ 62ea\ b250\ 1f61\ 78f0\ 99c4\ b813$$
$$s_{10}(t) = 2cb5\ d63f\ 257d\ 40d1\ f390\ a49a\ 14ee\ cf7a\ 99a8\ f1d7\ 7ea1\ 33c2\ 7120\ 23a9\ f832\ 4dc2$$
$$s_{11}(t) = 157c\ 965a\ eb93\ 9238\ 6138\ b9c9\ d6fd\ 1355\ e3ac\ cd40\ 7d85\ ef40\ c1d3\ b004\ 1394\ f6dd$$
$$s_{12}(t) = 3db7\ 4ba6\ 4a20\ f769\ 491f\ 39fc\ 6ea1\ e35d\ 8a82\ fb0a\ c680\ 8ea7\ e688\ 6469\ cd8a\ 11fa$$
$$s_{13}(t) = 06c7\ 9cc5\ a452\ a776\ 7bf4\ cd46\ 0eab\ e509\ fe17\ 8d01\ 1c4b\ 8192\ 6e41\ 5b59\ 61f6\ f5d1$$
$$s_{14}(t) = 39c9\ 4065\ ceee\ d2e9\ 92a8\ 1d53\ c213\ dc2f\ 7a04\ 3c97\ 0324\ dc82\ b0f3\ 93ad\ eba6\ bb1f.$$

Note that the first symbol in the hexadecimal representation of each sequence should be considered as an octal representation, whenever the sequence is expressed as a binary sequence.

Using the LCZ sequence set $\mathcal{S}_2$, we can construct a Butson Hadamard matrix with partially cyclic core

$$\mathcal{H}_{PC} = (h_{jk}),$$

where h_{jk} is

$$h_{jk} = \begin{cases} 1, & \text{if } j = 0 \text{ or } k = 0 \\ w^{s_{\lfloor (j-1)/L \rfloor}(k-1+j_L)}, & \text{otherwise} \end{cases}$$

and $L = (2^8 - 1)/(2^4 - 1) = 17$ and $j_L = (j - 1) \bmod L$. □

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