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On the error probability of quasi-orthogonal space–time block codes

Jae-Dong Yang[1,*,†], Xianglan Jin[1], Jong-Seon No[1] and Dong-Joon Shin[2]

[1] *Department of Electrical Engineering and Computer Science, INMC, Seoul National University, Seoul 151-744, Korea*
[2] *Department of Electronics and Computer Engineering, Hanyang University, Seoul 133-791, Korea*

SUMMARY

In this paper, we derive the exact pairwise error probabilities (PEPs) of various quasi-orthogonal space–time block codes (QO-STBCs) using the moment generating function. By classifying the exact PEPs of QO-STBCs into three types, we derive the closed-form expression for each type of PEP. Based on these closed-form expressions, we obtain the union bounds on the symbol error probability and bit error probability for QPSK modulation. Through simulation, it is shown that these union bounds are quite tight.

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1. INTRODUCTION

In a multiple-input multiple-output communication system, transmit diversity is a very attractive technique to achieve high reliability and spectral efficiency, and space–time code is a good solution to achieve transmit diversity. Tarokh *et al.* [1] introduced space–time codes and derived their performance criterion. Alamouti [2] introduced a simple transmit diversity scheme with two transmit antennas. Space–time block codes (STBCs) from orthogonal designs were proposed [3], which have low decoding complexity and achieve full diversity and full rate. Full rate means that one symbol is effectively transmitted during one symbol duration. However, it is known that the complex linear

*Correspondence to: Jae-Dong Yang, Department of Electrical Engineering and Computer Science, INMC, Seoul National University, Seoul 151-744, Korea.
†E-mail: yjdong@ccl.snu.ac.kr

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processing orthogonal design with full diversity and full rate exists only for two transmit antennas, which is Alamouti's scheme [2].

As it is not possible to construct the complex linear processing orthogonal STBC with full rate for more than two transmit antennas, Jafarkhani [4] introduced a quasi-orthogonal space–time block code (QO-STBC), which sacrifices full diversity to achieve full rate. Another QO-STBC was proposed by Tirkkonen *et al.* [5], which has the same performance and similar feature as the Jafarkhani scheme. Sharma and Papadias [6] showed that QO-STBCs can achieve full diversity by rotating signal constellation.

The performance analysis of space–time codes has been extensively studied. Taricco and Biglieri [7] derived the exact pairwise error probability (PEP) of space–time codes, and Lu *et al.* [8] analyzed the performance of space–time codes. Kim *et al.* [9] derived the exact symbol error probability (SEP) for the orthogonal STBC with quadrature amplitude modulation (QAM). Behnamfar *et al.* [10] derived tight upper and lower bounds on the bit error probability (BEP) and SEP of orthogonal STBC for arbitrary signaling schemes and mappings. Simon [11] analyzed the performance of space–time codes by deriving the exact PEP of space–time trellis codes (STTCs) based on the moment generating function (MGF), and Simon and Jafarkhani [12] derived the exact PEPs of super-quasi-orthogonal STTCs.

However, the SEP and BEP of QO-STBCs have not been analyzed. Thus, in this paper, the exact PEPs of QO-STBCs are derived by using the MGF. By classifying PEPs into three types, the closed-form expression for each type of PEP is derived. Using these results, union bounds on the SEP and BEP of various QO-STBCs for QPSK modulation are derived. It is also shown from the simulation that our union bounds are quite tight.

The organization of this paper is as follows. Various QO-STBCs are introduced in Section 2. In Section 3, the exact closed-form PEPs of QO-STBCs are derived. In Section 4, the union bounds on the SEP and BEP are evaluated, and in Section 5 we verify our analysis through simulation. In Section 6, conclusion is given.

2. QUASI-ORTHOGONAL STBCs

Consider a wireless communication system with L_t transmit antennas and L_r receive antennas. We consider square QO-STBCs. Let $\mathbf{X}$ be an $L_t \times L_t$ codeword matrix. From the first row to the last row, L_t symbols in a row of $\mathbf{X}$ are transmitted via transmit antennas simultaneously. We assume that the Rayleigh fading channel is quasi-static, i.e. the channel remains constant over the duration of a codeword matrix. Then the received signal can be expressed in the following matrix form:

$$\mathbf{Y} = \mathbf{XH} + \mathbf{N} \tag{1}$$

where $\mathbf{Y}$ is the $L_t \times L_r$ received signal matrix, $\mathbf{N}$ is the $L_t \times L_r$ additive white-Gaussian noise matrix, and $\mathbf{H}$ is the $L_t \times L_r$ channel coefficient matrix. The elements of $\mathbf{H}$ are independent complex Gaussian random variables with mean 0 and variance 0.5 per dimension. The elements of $\mathbf{N}$ are also independent complex Gaussian random variables with mean 0 and variance σ^2 per dimension.

A codeword matrix $\mathbf{X}$ of QO-STBC is an $L_t \times L_t$ matrix, which is not an orthogonal matrix, but some columns are orthogonal to each other. As the difference matrix between any two distinct codewords does not always have full rank, QO-STBCs cannot achieve full diversity. QO-STBCs have the decoding complexity higher than orthogonal STBCs but lower than STTCs. When the

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maximum likelihood (ML) decoding is performed, unlike orthogonal STBCs, a pair of symbols of 4×4 QO-STBCs are jointly decoded.

Jafarkhani proposed the 4×4 QO-STBC [4], which has the following codeword matrix:

$$\mathbf{X}=\begin{bmatrix} x_1 & x_2 & x_3 & x_4 \\ -x_2^* & x_1^* & -x_4^* & x_3^* \\ -x_3^* & -x_4^* & x_1^* & x_2^* \\ x_4 & -x_3 & -x_2 & x_1 \end{bmatrix} \tag{2}$$

Note that the first and the fourth columns are not orthogonal but they are orthogonal to the second and the third columns, which are not orthogonal to each other. As the difference matrix between two distinct codeword matrices does not guarantee full rank, this scheme cannot achieve full diversity.

Tirkkonen, Boariu, and Hottinen (TBH) also introduced the 4×4 QO-STBC [5], which has the following codeword matrix:

$$\mathbf{X}=\begin{bmatrix} x_1 & x_2 & x_3 & x_4 \\ -x_2^* & x_1^* & -x_4^* & x_3^* \\ x_3 & x_4 & x_1 & x_2 \\ -x_4^* & x_3^* & -x_2^* & x_1^* \end{bmatrix} \tag{3}$$

This scheme shows the same performance as Jafarkhani scheme and similarly does not guarantee full diversity.

As Jafarkhani and TBH schemes do not have full diversity, Sharma and Papadias (SP) proposed the constellation rotation technique to achieve full diversity in QO-STBCs [6]. The codeword matrix of SP scheme with rotation angle ϕ is given as

$$\mathbf{X}=\begin{bmatrix} x_1 & x_2 & \mathrm{e}^{\mathrm{j}\phi}x_3 & \mathrm{e}^{\mathrm{j}\phi}x_4 \\ x_2^* & -x_1^* & \mathrm{e}^{-\mathrm{j}\phi}x_4^* & -\mathrm{e}^{-\mathrm{j}\phi}x_3^* \\ \mathrm{e}^{\mathrm{j}\phi}x_3 & -\mathrm{e}^{\mathrm{j}\phi}x_4 & -x_1 & x_2 \\ \mathrm{e}^{-\mathrm{j}\phi}x_4^* & \mathrm{e}^{-\mathrm{j}\phi}x_3^* & -x_2^* & -x_1^* \end{bmatrix} \tag{4}$$

Without constellation rotation ($\phi=0$), codewords in (4) cannot have full diversity. However, if we rotate the constellation corresponding to the symbols x_3 and x_4, we can achieve full diversity. Su and Xia showed that for QPSK symbols, $\phi=\pi/4$ is the optimal value with respect to diversity product [13].

3. EXACT PAIRWISE ERROR PROBABILITY

In [11], the exact PEP of STTCs was derived using the MGF. We also use the MGF to evaluate the exact PEP of QO-STBCs.

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Let $\mathrm{vec}(\mathbf{A})$ denote the vectorization operator that stacks the columns of $\mathbf{A}$ and $\otimes$ denote the matrix Kronecker product. Then, (1) can be rewritten as

$$\mathbf{y} = (\mathbf{I}_{L_\mathrm{r}} \otimes \mathbf{X})\mathbf{h} + \mathbf{n}$$

where $\mathbf{y} = \mathrm{vec}(\mathbf{Y})$, $\mathbf{h} = \mathrm{vec}(\mathbf{H})$, $\mathbf{n} = \mathrm{vec}(\mathbf{N})$, and $\mathbf{I}_n$ is the $n \times n$ identity matrix.

Assume that the perfect channel state information is available at the receiver. Then, the ML decoding metric becomes

$$m(\mathbf{Y}, \mathbf{X}) = \|\mathbf{Y} - \mathbf{X}\mathbf{H}\|^2 = \|\mathbf{y} - (\mathbf{I}_{L_\mathrm{r}} \otimes \mathbf{X})\mathbf{h}\|^2 \tag{5}$$

The PEP conditioned on the channel, that is, the probability that the ML decoder decodes the correct $\mathbf{X}$ into incorrect $\hat{\mathbf{X}} \neq \mathbf{X}$, is given as

$$\begin{aligned} P(\mathbf{X} \to \hat{\mathbf{X}}|\mathbf{H}) &= \Pr\{m(\mathbf{Y}, \mathbf{X}) \geqslant m(\mathbf{Y}, \hat{\mathbf{X}})|\mathbf{H}\} \\ &= \Pr\{m(\mathbf{Y}, \mathbf{X}) - m(\mathbf{Y}, \hat{\mathbf{X}}) \geqslant 0|\mathbf{H}\} \end{aligned} \tag{6}$$

After substituting (5) into (6) and performing some calculations, we can obtain

$$P(\mathbf{X} \to \hat{\mathbf{X}}|\mathbf{H}) = \Pr\{z \geqslant \mu|\mathbf{H}\} = Q\left(\sqrt{\frac{\mu}{4\sigma^2}}\right)$$

where $\mu = \|[\mathbf{I}_{L_\mathrm{r}} \otimes (\mathbf{X} - \hat{\mathbf{X}})]\mathbf{h}\|^2$, $Q(\cdot)$ is the Q-function, and $z = 2Re\{\mathbf{n}^H[\mathbf{I}_{L_\mathrm{r}} \otimes (\hat{\mathbf{X}} - \mathbf{X})]\mathbf{h}\}$ is a conditionally zero mean real Gaussian random variable with variance $\sigma_z^2 = 4\sigma^2\mu$. Note that $(\cdot)^H$ denotes the complex conjugate transpose. If we normalize the average transmitted symbol energy from each antenna, i.e. $E\{|x_i|^2\} = 1$, then the noise variance σ^2 would be $L_\mathrm{t}/2\gamma$, where γ is the average signal-to-noise ratio (SNR). Using Craig's result [14], the conditional exact PEP is given as

$$P(\mathbf{X} \to \hat{\mathbf{X}}|\mathbf{H}) = \frac{1}{\pi}\int_0^{\pi/2} \exp\left[-\frac{\mu}{8\sigma^2 \sin^2\theta}\right] \mathrm{d}\theta \tag{7}$$

Substituting $\mu = \|[\mathbf{I}_{L_\mathrm{r}} \otimes (\mathbf{X} - \hat{\mathbf{X}})]\mathbf{h}\|^2$ and $\sigma^2 = L_\mathrm{t}/2\gamma$ into (7), we can obtain

$$P(\mathbf{X} \to \hat{\mathbf{X}}|\mathbf{H}) = \frac{1}{\pi}\int_0^{\pi/2} \exp\left[-\frac{\gamma}{4L_\mathrm{t}\sin^2\theta}\|[\mathbf{I}_{L_\mathrm{r}} \otimes (\mathbf{X} - \hat{\mathbf{X}})]\mathbf{h}\|^2\right] \mathrm{d}\theta \tag{8}$$

To evaluate the exact PEP, we need to average the conditional PEP in (8) over the channel matrix. As taking expectation over the channel matrix $\mathbf{H}$ is equivalent to taking expectation over the channel vector $\mathbf{h}$, we have

$$\begin{aligned} P(\mathbf{X} \to \hat{\mathbf{X}}) &= E_{\mathbf{H}}\{P(\mathbf{X} \to \hat{\mathbf{X}}|\mathbf{H})\} \\ &= \frac{1}{\pi}\int_0^{\pi/2} E_{\mathbf{h}}\left\{\exp\left[-\frac{\gamma}{4L_\mathrm{t}}\|[\mathbf{I}_{L_\mathrm{r}} \otimes (\mathbf{X} - \hat{\mathbf{X}})]\mathbf{h}\|^2 \frac{1}{\sin^2\theta}\right]\right\} \mathrm{d}\theta \\ &= \frac{1}{\pi}\int_0^{\pi/2} M_p\left(-\frac{1}{\sin^2\theta}\right) \mathrm{d}\theta \end{aligned}$$

where $p = (\gamma/4L_\mathrm{t})\mathbf{h}^H[\mathbf{I}_{L_\mathrm{r}} \otimes (\mathbf{X} - \hat{\mathbf{X}})]^H[\mathbf{I}_{L_\mathrm{r}} \otimes (\mathbf{X} - \hat{\mathbf{X}})]\mathbf{h}$ and $M_p(s) = E_p\{\exp(sp)\}$. Considering that p is the quadratic form of complex Gaussian random variables, we can use the MGF of the quadratic complex Gaussian random variables. Turin [15] derived the characteristic function of

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quadratic complex Gaussian random variables, and using this result the exact PEP can be obtained as [11]

$$
\begin{aligned}
P(\mathbf{X}\to\hat{\mathbf{X}}) &= \frac{1}{\pi}\int_0^{\pi/2}\left[\det\left(\mathbf{I}_{L_\mathrm{r}L_\mathrm{t}}+\frac{\gamma_s}{4\sin^2\theta}[\mathbf{I}_{L_\mathrm{r}}\otimes(\mathbf{X}-\hat{\mathbf{X}})]^H[\mathbf{I}_{L_\mathrm{r}}\otimes(\mathbf{X}-\hat{\mathbf{X}})]\right)\right]^{-1}\mathrm{d}\theta \\
&= \frac{1}{\pi}\int_0^{\pi/2}\left[\det\left(\mathbf{I}_{L_\mathrm{t}}+\frac{\gamma_s}{4\sin^2\theta}(\mathbf{X}-\hat{\mathbf{X}})^H(\mathbf{X}-\hat{\mathbf{X}})\right)\right]^{-L_\mathrm{r}}\mathrm{d}\theta \qquad (9)
\end{aligned}
$$

where $\gamma_s=\gamma/L_\mathrm{t}$. Using (9), we will find the closed-form expressions for the exact PEPs of various QO-STBCs as follows.

3.1. Jafarkhani and TBH schemes

To derive the exact PEP of the Jafarkhani scheme in (2), we have to calculate the determinant in (9) as

$$
\begin{aligned}
\det\left[\mathbf{I}_4+\frac{\gamma_s}{4\sin^2\theta}(\mathbf{X}-\hat{\mathbf{X}})^H(\mathbf{X}-\hat{\mathbf{X}})\right] &= \det\begin{bmatrix} 1+a & 0 & 0 & b \\ 0 & 1+a & -b & 0 \\ 0 & -b & 1+a & 0 \\ b & 0 & 0 & 1+a \end{bmatrix} \\
&= [(1+a)^2-b^2]^2 \qquad (10)
\end{aligned}
$$

where $a=(\gamma_s/4\sin^2\theta)\sum_{i=1}^{4}|x_i-\hat{x}_i|^2$ and $b=(\gamma_s/4\sin^2\theta)2Re\{(x_1-\hat{x}_1)(x_4-\hat{x}_4)^*-(x_2-\hat{x}_2)(x_3-\hat{x}_3)^*\}$. Substituting (10) into (9) and after some algebraic calculations, we can obtain the exact PEP of the Jafarkhani scheme. It is already known that the ML decoding of QO-STBC in (2) can be done pair by pair, i.e. x_1 and x_4 are jointly decoded and so are x_2 and x_3, independently [4]. Hence, we will consider only x_1 and x_4 to derive the PEP and use the notations $\mathbf{X}=(x_1,x_4)$ and $\hat{\mathbf{X}}=(\hat{x}_1,\hat{x}_4)$. Therefore, the exact PEP can be expressed as

$$
\begin{aligned}
P_J(\mathbf{X}\to\hat{\mathbf{X}}) = \frac{1}{\pi}\int_0^{\pi/2}&\left[1+\frac{\gamma_s}{4\sin^2\theta}|(x_1-\hat{x}_1)+(x_4-\hat{x}_4)|^2\right]^{-2L_\mathrm{r}} \\
&\times\left[1+\frac{\gamma_s}{4\sin^2\theta}|(x_1-\hat{x}_1)-(x_4-\hat{x}_4)|^2\right]^{-2L_\mathrm{r}}\mathrm{d}\theta \qquad (11)
\end{aligned}
$$

Similar to the Jafarkhani scheme, symbols x_1 and x_3 in (3) of TBH scheme are jointly decoded and so are x_2 and x_4, independently. To evaluate the exact PEP, we have to calculate the determinant in (9) as

$$
\begin{aligned}
\det\left[\mathbf{I}_4+\frac{\gamma_s}{4\sin^2\theta}(\mathbf{X}-\hat{\mathbf{X}})^H(\mathbf{X}-\hat{\mathbf{X}})\right] &= \det\begin{bmatrix} 1+a & 0 & b & 0 \\ 0 & 1+a & 0 & b \\ b & 0 & 1+a & 0 \\ 0 & b & 0 & 1+a \end{bmatrix} \\
&= [(1+a)^2-b^2]^2
\end{aligned}
$$

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where a is the same as that in (10) and $b=(\gamma_s/4\sin^2\theta)2Re\{(x_1-\hat{x}_1)(x_3-\hat{x}_3)^*+(x_2-\hat{x}_2)(x_4-\hat{x}_4)^*\}$. Using the same method as the Jafarkhani scheme, we can derive the PEP of the TBH scheme as

$$P_{\mathrm{TBH}}(\mathbf{X}\to\hat{\mathbf{X}})=\frac{1}{\pi}\int_0^{\pi/2}\left[1+\frac{\gamma_s}{4\sin^2\theta}|(x_1-\hat{x}_1)+(x_3-\hat{x}_3)|^2\right]^{-2L_\mathrm{r}} \times\left[1+\frac{\gamma_s}{4\sin^2\theta}|(x_1-\hat{x}_1)-(x_3-\hat{x}_3)|^2\right]^{-2L_\mathrm{r}}\mathrm{d}\theta \tag{12}$$

3.2. SP scheme

In order to find the exact PEP of the SP scheme in (4), we have to calculate the determinant in (9) as

$$\det\left[\mathbf{I}_4+\frac{\gamma_s}{4\sin^2\theta}(\mathbf{X}-\hat{\mathbf{X}})^H(\mathbf{X}-\hat{\mathbf{X}})\right]=\det\begin{bmatrix}1+a & 0 & \mathrm{j}b & 0\\ 0 & 1+a & 0 & -\mathrm{j}b\\ -\mathrm{j}b & 0 & 1+a & 0\\ 0 & \mathrm{j}b & 0 & 1+a\end{bmatrix}$$
$$=[(1+a)^2-b^2]^2$$

where a is the same as that in (10) and $b=(\gamma_s/4\sin^2\theta)2\,\mathrm{Im}\{(x_1-\hat{x}_1)^*(x_3-\hat{x}_3)\mathrm{e}^{\mathrm{j}\phi}+(x_2-\hat{x}_2)(x_4-\hat{x}_4)^*\mathrm{e}^{-\mathrm{j}\phi}\}$. We can choose ϕ (e.g. $\pi/4$) to achieve full diversity [13]. After some algebraic calculations, we can obtain the exact PEP using only x_1 and x_3 as

$$P_{\mathrm{SP}}(\mathbf{X}\to\hat{\mathbf{X}})=\frac{1}{\pi}\int_0^{\pi/2}\left[1+\frac{\gamma_s}{4\sin^2\theta}|(x_1-\hat{x}_1)+\mathrm{j}\mathrm{e}^{\mathrm{j}\phi}(x_3-\hat{x}_3)|^2\right]^{-2L_\mathrm{r}} \times\left[1+\frac{\gamma_s}{4\sin^2\theta}|(x_1-\hat{x}_1)-\mathrm{j}\mathrm{e}^{\mathrm{j}\phi}(x_3-\hat{x}_3)|^2\right]^{-2L_\mathrm{r}}\mathrm{d}\theta \tag{13}$$

Note that (13) has the similar expression as (11) and (12).

3.3. Closed-form expressions

Now, we obtain the exact PEPs of three different schemes of QO-STBCs. As (11), (12), and (13) are similar, we can unify the exact PEPs by using u and v defined in Table I. For example, in the Jafarkhani scheme, u is equal to $|(x_1-\hat{x}_1)+(x_4-\hat{x}_4)|^2$ and v is equal to $|(x_1-\hat{x}_1)-(x_4-\hat{x}_4)|^2$. By using u and v in Table I, (11), (12), and (13) can be commonly expressed as the following one equation:

$$P(\mathbf{X}\to\hat{\mathbf{X}})=\frac{1}{\pi}\int_0^{\pi/2}\left(\frac{\sin^2\theta}{\sin^2\theta+(\gamma_s/4)u}\right)^{2L_\mathrm{r}}\left(\frac{\sin^2\theta}{\sin^2\theta+(\gamma_s/4)v}\right)^{2L_\mathrm{r}}\mathrm{d}\theta \tag{14}$$

Depending on u and v, the exact PEP (14) can be classified into three types. Note that as we are considering PEP, both u and v cannot be zero.

Table I. u and v for various QO-STBCs.

	u	v
J	$\|(x_1-\hat{x}_1)+(x_4-\hat{x}_4)\|^2$	$\|(x_1-\hat{x}_1)-(x_4-\hat{x}_4)\|^2$
TBH	$\|(x_1-\hat{x}_1)+(x_3-\hat{x}_3)\|^2$	$\|(x_1-\hat{x}_1)-(x_3-\hat{x}_3)\|^2$
SP	$\|(x_1-\hat{x}_1)+\mathrm{j}e^{\mathrm{j}\phi}(x_3-\hat{x}_3)\|^2$	$\|(x_1-\hat{x}_1)-\mathrm{j}e^{\mathrm{j}\phi}(x_3-\hat{x}_3)\|^2$

Table II. Distribution of u and v for the Jafarkhani scheme with QPSK.

Type	u	v	s_1	s_2	b_1	b_2	b_3	b_4
I	0	8	0	16	0	16	0	0
	0	16	0	4	0	0	0	4
	8	0	0	16	0	16	0	0
	16	0	0	4	0	0	0	4
II	2	2	64	0	64	0	0	0
	4	4	32	32	0	64	0	0
	8	8	0	8	0	0	0	8
III	2	10	0	32	0	0	32	0
	10	2	0	32	0	0	32	0

s_i, number of i-symbol-error cases; b_i, number of i-bit-error cases.

Type I is the case when only one of u and v is equal to zero. In this case, there are two symbol errors between $\mathbf{X}$ and $\hat{\mathbf{X}}$. As the exact PEP of Type I, (14) can be rewritten as

$$P_{\mathrm{I}}(\mathbf{X}\rightarrow\hat{\mathbf{X}})=\frac{1}{\pi}\int_0^{\pi/2}\left(\frac{\sin^2\theta}{\sin^2\theta+(\gamma_s/4)(u+v)}\right)^{2L_{\mathrm{r}}}\mathrm{d}\theta \tag{15}$$

Type II is the case of $u=v\neq 0$. In this case, there are one or two symbol errors between $\mathbf{X}$ and $\hat{\mathbf{X}}$. As the exact PEP of Type II, (14) can be rewritten as

$$P_{\mathrm{II}}(\mathbf{X}\rightarrow\hat{\mathbf{X}})=\frac{1}{\pi}\int_0^{\pi/2}\left(\frac{\sin^2\theta}{\sin^2\theta+(\gamma_s/4)u}\right)^{4L_{\mathrm{r}}}\mathrm{d}\theta$$

Type III is the case when u and v are nonzero and distinct. In this case, there are two symbol errors between $\mathbf{X}$ and $\hat{\mathbf{X}}$, and the exact PEP of Type III has the same expression as (14), that is,

$$P_{\mathrm{III}}(\mathbf{X}\rightarrow\hat{\mathbf{X}})=P(\mathbf{X}\rightarrow\hat{\mathbf{X}})$$

It is easy to see that when there is only one symbol error, the exact PEP is classified into Type II. However, Type II also includes two symbol errors. From Tables II and III, we can see the relationship between the type of PEP and the number of i-symbol-error cases, where i-symbol-error means that there are i symbol errors between $\mathbf{X}$ and $\hat{\mathbf{X}}$. From (15), we can see that QO-STBC does not achieve full diversity. Unlike the Jafarkhani and TBH schemes, the SP scheme with constellation rotation does not have Type I and thus has full diversity.

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Table III. Distribution of u and v for the SP scheme with QPSK.

Type	u	v	s_1	s_2	b_1	b_2	b_3	b_4
II	2	2	64	0	64	0	0	0
	4	4	32	0	0	32	0	0
	6	6	0	32	0	0	32	0
III	0.343	11.657	0	16	0	0	16	0
	1.172	6.828	0	32	0	32	0	0
	2.343	13.657	0	8	0	0	0	8
	6.828	1.172	0	32	0	32	0	0
	11.657	0.343	0	16	0	0	16	0
	13.657	2.343	0	8	0	0	0	8

s_i, number of i-symbol-error cases; b_i, number of i-bit-error cases.

Using the results in [16, Appendix 5A], we can derive the closed-form expressions of the three types of exact PEPs for QO-STBCs as

$$P_{\mathrm{I}}(\mathbf{X}\to\hat{\mathbf{X}})=\frac{1}{2}\left[1-\sqrt{\frac{\gamma_s(u+v)}{4+\gamma_s(u+v)}}\sum_{k=0}^{2L_{\mathrm{r}}-1}\binom{2k}{k}\left(\frac{1}{4+\gamma_s(u+v)}\right)^k\right]$$

$$P_{\mathrm{II}}(\mathbf{X}\to\hat{\mathbf{X}})=\frac{1}{2}\left[1-\sqrt{\frac{\gamma_s u}{4+\gamma_s u}}\sum_{k=0}^{4L_{\mathrm{r}}-1}\binom{2k}{k}\left(\frac{1}{4+\gamma_s u}\right)^k\right]$$

$$P_{\mathrm{III}}(\mathbf{X}\to\hat{\mathbf{X}})=\frac{(u/v)^{2L_{\mathrm{r}}-1}}{2(1-u/v)^{4L_{\mathrm{r}}-1}}\left[\sum_{k=0}^{2L_{\mathrm{r}}-1}\left(\frac{v}{u}-1\right)^k B_k I_k\left(\frac{\gamma_s}{4}v\right)\right.$$
$$\left.-\frac{u}{v}\sum_{k=0}^{2L_{\mathrm{r}}-1}\left(1-\frac{u}{v}\right)^k C_k I_k\left(\frac{\gamma_s}{4}u\right)\right]$$

where

$$A_k=(-1)^{2L_{\mathrm{r}}-1+k}\frac{\binom{2L_{\mathrm{r}}-1}{k}}{(2L_{\mathrm{r}}-1)!}\prod_{\substack{n=1\\ n\neq k+1}}^{2L_{\mathrm{r}}}(4L_{\mathrm{r}}-n)$$

$$B_k=\frac{A_k}{\binom{4L_{\mathrm{r}}-1}{k}},\quad C_k=\sum_{n=0}^{2L_{\mathrm{r}}-1}\frac{\binom{k}{n}}{\binom{4L_{\mathrm{r}}-1}{n}}A_n$$

and

$$I_k(c)=1-\sqrt{\frac{c}{1+c}}\left[1+\sum_{n=1}^{k}\frac{(2n-1)!!}{n!2^n(1+c)^n}\right]$$

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The double factorial $(2n-1)!!$ denotes the product of only odd integers from 1 to $2n-1$ and note that $\binom{k}{n}=0$ for $n>k$.

3.4. QO-STBCs for more than four transmit antennas

QO-STBCs for more than four transmit antennas can be constructed by using two orthogonal STBCs with the same structure. Let $\mathbf{A}$ and $\mathbf{B}$ be two $N\times N$ orthogonal STBCs consisting of $x_1,\ldots,x_T$ and $x_{T+1},\ldots,x_{2T}$, respectively. By using $\mathbf{A}$ and $\mathbf{B}$, a $2N\times 2N$ QO-STBC is constructed as

$$\mathbf{X}=\begin{bmatrix}\mathbf{A} & \mathbf{B}\\ \mathbf{B} & \mathbf{A}\end{bmatrix} \tag{16}$$

As this construction method is similar to the 4×4 TBH scheme, the exact PEP can be obtained similarly as in the previous subsections. We need to calculate only the determinant in (9) as [13]

$$\det\left[\mathbf{I}_{2N}+\frac{\gamma_s}{4\sin^2\theta}(\mathbf{X}-\hat{\mathbf{X}})^H(\mathbf{X}-\hat{\mathbf{X}})\right]=\det\begin{bmatrix}(1+a)\mathbf{I}_N & b\mathbf{I}_N\\ b\mathbf{I}_N & (1+a)\mathbf{I}_N\end{bmatrix}=[(1+a)^2-b^2]^N$$

where $a=(\gamma_s/4\sin^2\theta)\sum_{i=1}^{2T}|x_i-\hat{x}_i|^2$ and $b=(\gamma_s/4\sin^2\theta)2Re\{\sum_{i=1}^{T}(x_i-\hat{x}_i)(x_{T+i}-\hat{x}_{T+i})^*\}$.

The ML decoding of (16) is done similarly as the TBH scheme. Thus, we consider only x_1 and x_{T+1} to derive the exact PEP. The exact PEP of (16) is given as

$$\begin{aligned}P(\mathbf{X}\to\hat{\mathbf{X}})=\frac{1}{\pi}\int_0^{\pi/2}&\left[1+\frac{\gamma_s}{4\sin^2\theta}|(x_1-\hat{x}_1)+(x_{T+1}-\hat{x}_{T+1})|^2\right]^{-NL_\mathrm{r}}\\&\times\left[1+\frac{\gamma_s}{4\sin^2\theta}|(x_1-\hat{x}_1)-(x_{T+1}-\hat{x}_{T+1})|^2\right]^{-NL_\mathrm{r}}\mathrm{d}\theta\end{aligned} \tag{17}$$

The difference between (12) and (17) is the exponent in the integrand. Thus, the closed-form expression for the exact PEP can be obtained by replacing $2L_\mathrm{r}$ with NL_r and $4L_\mathrm{r}$ with $2NL_\mathrm{r}$.

4. UNION BOUNDS ON THE SEP AND BEP

In this section, we derive the union bound on the SEP of QO-STBCs using the exact PEPs as

$$\mathrm{SEP}\leqslant\frac{1}{n_s}\sum_{\mathbf{X}}\left[\sum_{\hat{\mathbf{X}}\neq\mathbf{X}}P(\mathbf{X}\to\hat{\mathbf{X}})d_{SH}(\mathbf{X},\hat{\mathbf{X}})\right]p(\mathbf{X})$$

where n_s is the number of symbols of $\mathbf{X}$ in PEP expression, $d_{SH}(\mathbf{X},\hat{\mathbf{X}})$ is the number of different symbols between $\mathbf{X}$ and $\hat{\mathbf{X}}$, and $p(\mathbf{X})$ is the probability that $\mathbf{X}$ is transmitted. Similarly, we also derive the union bound on the BEP of QO-STBCs using the exact PEPs as

$$\mathrm{BEP}\leqslant\frac{1}{n_b}\sum_{\mathbf{X}}\left[\sum_{\hat{\mathbf{X}}\neq\mathbf{X}}P(\mathbf{X}\to\hat{\mathbf{X}})d_{BH}(\mathbf{X},\hat{\mathbf{X}})\right]p(\mathbf{X})$$

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where n_b is the number of bits of $\mathbf{X}$ and $d_{BH}(\mathbf{X},\hat{\mathbf{X}})$ is the number of different bits between $\mathbf{X}$ and $\hat{\mathbf{X}}$.

We will derive the union bounds on the SEP and the BEP of a 4×4 QO-STBC in (2). The union bounds on the SEP and the BEP of 4×4 QO-STBCs in (3) and (4) can be derived in the same manner. We consider QPSK modulation with Gray mapping, i.e. $x_i \in \{e^{j(2\pi k/4+\pi/4)}|0\leqslant k\leqslant 3\}$. We assume L_r receive antennas. These can be similarly applied to other modulation schemes. We assume that the information symbols are equiprobable. As only x_1 and x_4 appear in (11), the union bounds are given as

$$\begin{aligned}\mathrm{SEP} &\leqslant \frac{1}{2}\sum_{x_1,x_4}\left[\sum_{(\hat{x}_1,\hat{x}_4)\neq(x_1,x_4)} P_{14}(\mathbf{X}\rightarrow\hat{\mathbf{X}})d_{SH}(\mathbf{X},\hat{\mathbf{X}})\right]p(\mathbf{X})\\ &\leqslant \frac{1}{32}\sum_{x_1,x_4}\left[\sum_{(\hat{x}_1,\hat{x}_4)\neq(x_1,x_4)} P_{14}(\mathbf{X}\rightarrow\hat{\mathbf{X}})d_{SH}(\mathbf{X},\hat{\mathbf{X}})\right]\end{aligned} \tag{18}$$

$$\begin{aligned}\mathrm{BEP} &\leqslant \frac{1}{4}\sum_{x_1,x_4}\left[\sum_{(\hat{x}_1,\hat{x}_4)\neq(x_1,x_4)} P_{14}(\mathbf{X}\rightarrow\hat{\mathbf{X}})d_{BH}(\mathbf{X},\hat{\mathbf{X}})\right]p(\mathbf{X})\\ &\leqslant \frac{1}{64}\sum_{x_1,x_4}\left[\sum_{(\hat{x}_1,\hat{x}_4)\neq(x_1,x_4)} P_{14}(\mathbf{X}\rightarrow\hat{\mathbf{X}})d_{BH}(\mathbf{X},\hat{\mathbf{X}})\right]\end{aligned} \tag{19}$$

where P_{14} means the PEP for only x_1 and x_4 of $\mathbf{X}$. Tables II and III show the distribution of u and v and corresponding PEP types. They also show the number of i-symbol-error cases and the number of i-bit-error cases. Let s_i denote the number of i-symbol-error cases and b_i the number of i-bit-error cases. It is clear that $\sum_{i=1}^{2} s_i = \sum_{i=1}^{i=4} b_i$ for a fixed (u,v) pair. Tables II and III correspond to the Jafarkhani and SP schemes, respectively. It can be easily shown that the TBH scheme has the same (u,v) distribution as that of the Jafarkhani scheme. When we apply QPSK modulation, there are 240 possible (u,v) pairs. We can calculate the exact PEP for each (u,v) pair. By using the number of symbol errors and bit errors for each (u,v) pair in Tables II and III, the union bounds on the SEP and the BEP can be evaluated from (18) and (19).

5. NUMERICAL RESULTS

For numerical analysis, four transmit antennas and $L_r = 1,2,4$ receive antennas are used and the optimal ML detection is performed. In addition, independent quasi-static Rayleigh fading channel is assumed, which means that the fading coefficients are constant over a codeword block and change independently block by block.

Figure 1 compares the union bound on the SEP and the simulation result, where 'UB' means the union bound on the SEP, 'Simulation' means the SEP obtained by simulation, 'J' means the Jafarkhani scheme, and 'SP' means the Sharma and Papadias scheme. Figure 2 compares the union bound on the BEP and the simulation result. Note that the Jafarkhani and TBH schemes have the same performance and thus only the Jafarkhani scheme is considered. In a low SNR region, the

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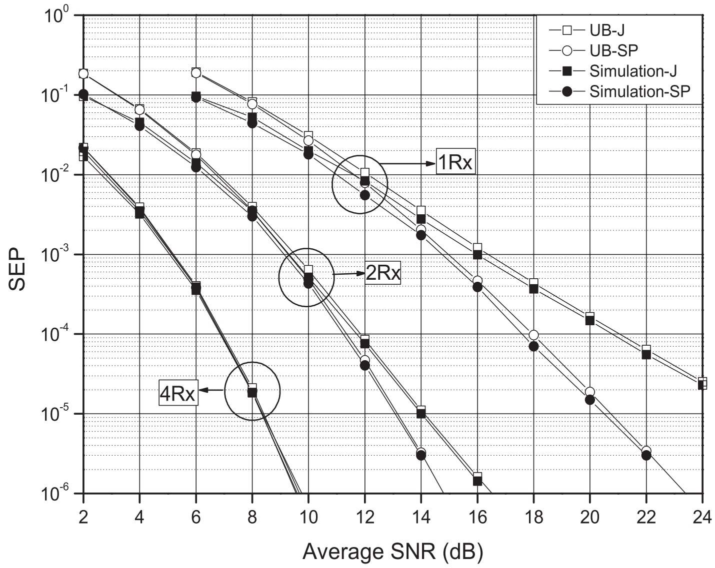

Figure 1. Union bounds on the SEP and simulation results for QO-STBCs with QPSK.

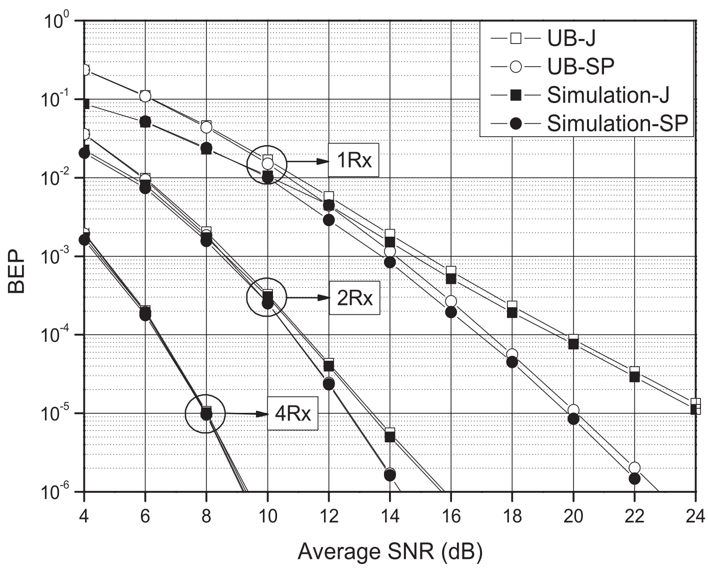

Figure 2. Union bounds on the BEP and simulation results for QO-STBCs with QPSK.

union bounds are loose, whereas in a high SNR region, they become tight. However, as the number of receive antenna increases, the union bound becomes tight. Figures 1 and 2 also show that the SP scheme achieves full diversity contrary to the Jafarkhani scheme.

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6. CONCLUSION

We derived the exact closed-form expressions for PEPs of various QO-STBCs using the MGF by classifying the PEPs into three types. We also derived the union bounds on the SEP and the BEP of QO-STBC with QPSK modulation, which turn out to be tight. For high-order modulations such as MPSK and MQAM, the same analysis can be applied. We need to calculate only the values of u and v, and the number of i-symbol-error cases and i-bit-error cases corresponding to each (u, v), i.e. s_i and b_i similarly as in Table II for QPSK.

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AUTHORS' BIOGRAPHIES

Jae-Dong Yang received the BS degree in electronics engineering from Seoul National University, Seoul, Korea, in 2002. Since 2003, he is a PhD candidate at the Department of Electrical Engineering and Computer Science in Seoul National University, where he is pursuing his MS/PhD joint course. His research area includes space–time codes, MIMO, coding theory, information theory, and wireless communication systems.

Xianglan Jin received the BS and MS degrees in telecommunication engineering from Beijing University of Posts and Telecommunications, Beijing, China, in 1999 and 2002, respectively. She was an engineer at Beijing Samsung Telecommunications R&D center from April 2002 to June 2004. Currently, she is a PhD candidate in the Department of Electrical Engineering and Computer Science, Seoul National University. Her research interests include MIMO, space–time codes, and information theory.

Jong-Seon No received the BS and MSEE degrees in electronics engineering from Seoul National University, Seoul, Korea, in 1981 and 1984, respectively, and the PhD degree in electrical engineering from the University of Southern California, Los Angeles, in 1988. He was a Senior MTS at Hughes Network Systems from February 1988 to July 1990. He was also an Associate Professor in the Department of Electronic Engineering, Konkuk University, Seoul, Korea, from September 1990 to July 1999. He joined the faculty of the Department of Electrical Engineering and Computer Science, Seoul National University, in August 1999, where he is currently a Professor. His area of research interests include error-correcting codes, sequences, cryptography, space–times codes, LDPC codes, and wireless communication systems.

Dong-Joon Shin received the BS degree in electronics engineering from Seoul National University, Seoul, Korea, the MS degree in electrical engineering from Northwestern University, Evanston, U.S.A., and the PhD degree in electrical engineering from University of Southern California, Los Angeles, U.S.A. From 1999 to 2000, he was a member of the technical staff in Wireless Network Division and Satellite Network Division, Hughes Network Systems, MD, U.S.A. Since September 2000, he has been an Associate Professor in the Department of ECE at Hanyang University, Seoul, Korea. His current research interests include digital communication systems, error-correcting codes, sequences, and discrete mathematics.

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