LETTER

Convergence Speed Analysis of Layered Decoding of Block-Type LDPC Codes

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SUMMARY In this letter, we analyze the convergence speed of layered decoding of block-type low-density parity-check codes and verify that the layered decoding gives faster convergence speed than the sequential decoding with randomly selected check node subsets. Also, it is shown that using more subsets than the maximum variable node degree does not improve the convergence speed.

key words: *block-type low-density parity-check (B-LDPC) codes, convergence speed, density evolution, layered decoding*

1. Introduction

The layered decoding [1] has been proposed to improve the decoding and hardware implementation efficiency of low-density parity-check (LDPC) codes [1], [2]. Moreover, the convergence speed improvement of layered decoding is verified only by the numerical experiments.

For the sequential decoding by partitioning check nodes [3], [4], the check nodes of LDPC codes are partitioned into p subsets. For the first subset, each variable node sends the updated messages to its neighboring check nodes within the first subset. Then, the check nodes in that subset update and send the messages to their neighboring variable nodes. This procedure is sequentially carried out for the remaining $p-1$ subsets of check nodes, which makes one iteration. For the layered decoding, the check nodes of LDPC code are partitioned into subsets such that each variable node connected by check nodes within a subset has at most a single connection. The decoding procedure is the same as that for the sequential decoding.

In this letter, the layered decoding of block-type LDPC (B-LDPC) codes is analyzed using the results in [3]. Based on these results, it is shown that the layered decoding gives faster convergence speed than the other horizontal sequential decoding [4], which is confirmed by the simulation results. Also, it is shown that using more subsets of check nodes than the maximum variable node degree does not improve the convergence speed of layered decoding.

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2. Density Evolution Analysis with a Gaussian Approximation

In this section, we analyze the convergence speed of the layered decoding using the results in [3] by considering (d_v, d_c) regular LDPC codes and irregular LDPC codes, where d_v and d_c are the variable node and check node degrees, respectively.

Let v and u be the messages in the log likelihood ratio (LLR) form from variable node to check node and vice versa, respectively. Also, let m_v and m_u be the means of v and u, respectively, and $\phi(x)$ the function defined by

$$\phi(x) = 1 - \frac{1}{\sqrt{4\pi x}} \int_R \tanh\frac{u}{2} e^{-\frac{(u-x)^2}{4x}} du, \quad \text{for } x > 0 \qquad (1)$$

where $\phi(0) = 1$ and $\phi(\infty) = 0$.

2.1 Density Evolution Analysis of Layered Decoding

The number of distinct edge distributions for the connections from a variable node to p subsets of check nodes is ${}_p\mathrm{C}_{d_v} = p!/d_v!(p-d_v)!$, where $x! = x\times(x-1)\times\cdots\times 2\times 1$. If one edge from a variable node is connected to the subset S_i, the number of distinct edge distributions becomes ${}_{p-1}\mathrm{C}_{d_v-1}$. Let $(a_1, a_2, \cdots, a_p)$ denote the edge distribution of a variable node where a_j is the number of edges connected from a variable node to the check nodes in S_j, a_j takes zero or one, and $\sum_{j=1}^{p} a_j = d_v$. Then, the mean of message from this variable node to the subset S_i at the l-th iteration can be expressed as

$$m_{u_0} + \sum_{j=1}^{i-1} a_j m_{u_{S_j}}^{(l)} + \sum_{j=i+1}^{p} a_j m_{u_{S_j}}^{(l-1)} \qquad (2)$$

where m_{u_0} denotes the mean of the received message from the channel, u_{S_j} the message from the check node in S_j, and l the iteration number. The probability that the message with the mean value in (2) is passed to the subset S_i is $1/{}_{p-1}\mathrm{C}_{d_v-1}$. The recursive equation of the mean of the message from the check node in S_i to variable node at the l-th iteration can be expressed as

$$m_{u_{S_i}}^{(l)} = \phi^{-1}\left(1 - \left[1 - \frac{1}{{}_{p-1}\mathrm{C}_{d_v-1}} \sum_{(a_1,\cdots,a_p)} \phi\left(m_{u_0}\right.\right.\right.$$

$$+\sum_{j=1}^{i-1} a_j m_{u_{S_j}}^{(l)} + \sum_{j=i+1}^{p} a_j m_{u_{S_j}}^{(l-1)}\Bigg)\Bigg]^{d_c-1}\Bigg). \tag{3}$$

Similarly, for irregular LDPC codes, we can obtain

$$m_{u_{S_i}}^{(l)} = \sum_{r=2}^{d_r} \rho_r \phi^{-1}\Bigg(1 - \Bigg[1 - \sum_{q=2}^{d_l} \frac{\lambda_q}{{}_{p-1}\mathrm{C}_{q-1}} \sum_{(a_1,\cdots,a_p)} \phi\Bigg(m_{u_0} + \sum_{j=1}^{i-1} a_j m_{u_{S_j}}^{(l)} + \sum_{j=i+1}^{p} a_j m_{u_{S_j}}^{(l-1)}\Bigg)\Bigg]^{r-1}\Bigg) \tag{4}$$

where λ_i (ρ_j) is the fraction of edges with degree i (j) in terms of variable (check) node and d_l (d_r) stands for the maximum variable (check) node degree.

If all-zero codeword is transmitted, the bit error rate (BER) of codeword bits can be derived as

$$P_e = \Pr(x < 0) = \frac{1}{{}_p\mathrm{C}_{d_v}} \sum_{k=1}^{{}_p\mathrm{C}_{d_v}} Q\left(\sqrt{\frac{m_k^{(l)}}{2}}\right) \tag{5}$$

where $m_k^{(l)} = m_{u_0} + \sum_{j=1}^{p} a_j m_{u_{S_j}}^{(l)}$ and $Q(x) = \frac{1}{\sqrt{2\pi}} \int_x^{\infty} \exp\left(-\frac{u^2}{2}\right) du$.

Also, for irregular LDPC codes, we can derive

$$P_e = \sum_{q=2}^{d_l} \frac{\lambda'_q}{{}_p\mathrm{C}_q} \sum_{k=1}^{{}_p\mathrm{C}_q} Q\left(\sqrt{\frac{m_k^{(l)}}{2}}\right) \tag{6}$$

where λ'_i is the fraction of variable nodes with degree i.

2.2 Density Evolution Analysis of Sequential Decoding with Random Partitioning

For the sequential decoding algorithm with randomly selected p subsets of check nodes, the recursive equation and the BER can be expressed as follows.

$$m_{u_{S_i}}^{(l)} = \phi^{-1}\Bigg(1 - \Bigg[1 - \sum_{\substack{(a_1,\cdots,a_p)\\ a_i \neq 0}} \frac{(d_v-1)!}{a_1!\cdots(a_i-1)!\cdots a_p!\,p^{d_v-1}} \times \phi\Bigg(m_{u_0} + (a_i-1)m_{u_{S_i}}^{(l-1)} + \sum_{j=1}^{i-1} a_j m_{u_{S_j}}^{(l)} + \sum_{j=i+1}^{p} a_j m_{u_{S_j}}^{(l-1)}\Bigg)\Bigg]^{d_c-1}\Bigg). \tag{7}$$

$$P_e = \sum_{(a_1,\cdots,a_p)} p_{(a_1,\cdots,a_p)}\, Q\left(\sqrt{\frac{m_{(a_1,\cdots,a_p)}^{(l)}}{2}}\right) \tag{8}$$

where $p_{(a_1,\cdots,a_p)} = d_v!/(a_1!a_2!\cdots a_p!p^{d_v})$ and $m_{(a_1,\cdots,a_p)}^{(l)} = m_{u_0} + \sum_{j=1}^{p} a_j m_{u_{S_j}}^{(l)}$.

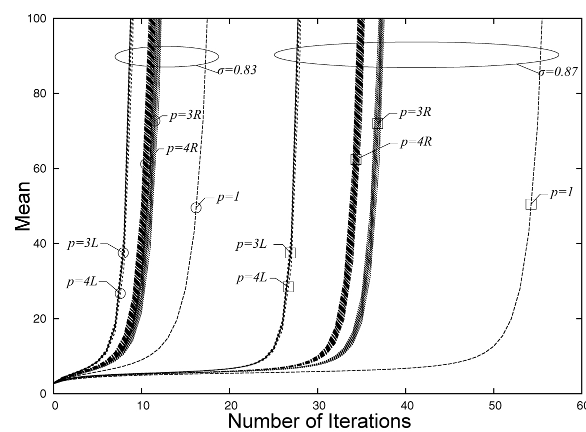
Similarly, for irregular LDPC codes, the recursive equation and the BER can be derived as follows.

$$m_{u_{S_i}}^{(l)} = \sum_{r=2}^{d_r} \rho_r \phi^{-1}\Bigg(1 - \Bigg[1 - \sum_{q=2}^{d_l} \lambda_q \sum_{\substack{(a_1,\cdots,a_p)\\ a_i \neq 0}} (q-1)! \times \frac{1}{a_1!\cdots(a_i-1)!\cdots a_p!\,p^{q-1}} \phi\Bigg(m_{u_0} + (a_i-1) \times m_{u_{S_i}}^{(l-1)} + \sum_{j=1}^{i-1} a_j m_{u_{S_j}}^{(l)} + \sum_{j=i+1}^{p} a_j m_{u_{S_j}}^{(l-1)}\Bigg)\Bigg]^{r-1}\Bigg). \tag{9}$$

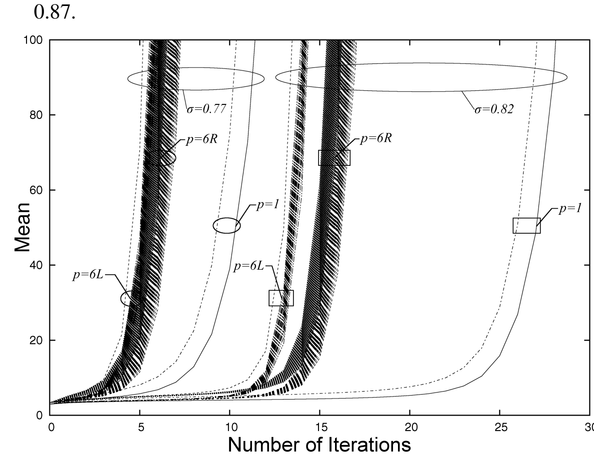
$$P_e = \sum_{q=2}^{d_l} \lambda'_q \sum_{(a_1,\cdots,a_p)} \frac{q!}{(a_1!\,a_2!\cdots a_p!\,p^q)} Q\left(\sqrt{\frac{m_{(a_1,\cdots,a_p)}^{(l)}}{2}}\right). \tag{10}$$

2.3 Mean Evolutions

Using (3) and (7), the mean evolution of (d_v, d_c) regular LDPC code at each iteration can be obtained. Equation (1) can be simplified as $\phi(x) \approx e^{-0.4527x^{0.86}+0.0218}$. We consider binary phase-shift keying (BPSK) modulation with magnitude 1 and the additive white Gaussian noise (AWGN) channel with mean 0 and variance σ^2.



(a) (3,6) regular LDPC code in the AWGN channel with $\sigma = 0.83$ and 0.87.



(b) Irregular LDPC code in the AWGN channel with $\sigma = 0.77$ and 0.82.

Fig. 1 Mean evolutions for the layered decoding and the sequential decoding with random partitioning for various p.

Figure 1(a) shows the mean evolution of $(3, 6)$ regular LDPC code for the various decodings with $p = 3$ and 4, respectively. We consider the AWGN channel with $\sigma = 0.83$ and 0.87 which are less than the threshold 0.8747. L and R stand for the layered decoding and the random partitioning, respectively. Figure 1(a) shows that as p increases, the performance improves and the layered decoding converges faster than the sequential decoding with random partitioning and the conventional decoding with flooding scheduling ($p = 1$). Also, Fig. 1(a) shows that the gain in the convergence speed becomes negligible for $p \geq 4$. If p goes beyond the maximum variable node degree $d_{\max}$, the overall layered decoding has similar structure to that in the case of $p = d_{\max}$ and the performance becomes similar.

Similarly, using (4) and (9), the mean evolution of irregular LDPC codes at each iteration can be obtained. For the analysis, a random ensemble of irregular LDPC codes with variable node degree distribution $\lambda(x) = 0.333333x^2 + 0.666667x^5$ and check node degree distribution $\rho(x) = x^8$ is used, where $\lambda(x) = \sum_{q=2}^{d_l} \lambda_q x^{q-1}$ and $\rho(x) = \sum_{r=2}^{d_r} \rho_r x^{r-1}$. Figure 1(b) shows the mean evolution of irregular LDPC code for the various decodings with $p = 6$. For the mean evolutions of irregular case, two patterns of mean evolutions corresponding to degree-3 and degree-6 variable nodes are observed for each decoding. The higher the degree of variable nodes is, the faster the convergence speed is. Also, from Fig. 1(b), we can see that the irregular case shows the similar trend of the regular case.

3. Simulation Results and Conclusions

For the simulation, we assume the BPSK modulation, AWGN channel, and the maximum number of iterations = 50. Figure 2 shows the 12×24 model matrices of $(3, 6)$ regular B-LDPC code and irregular B-LDPC code of rate $1/2$ with girth (shortest cycle) 10 and 6, respectively. Each blank block denotes 4177×4177 all-zero matrix and each block with a shift value represents a circularly right shifted 4177×4177 identity matrix by that value. For the regular B-LDPC code in Fig. 2(a), three subsets of check nodes, $\{1, 2, 3, 4\}$, $\{5, 6, 7, 8\}$, and $\{9, 10, 11, 12\}$-th block rows, are used for the layered decoding with $p = 3$. For the irregular B-LDPC code in Fig. 2(b), six subsets of check nodes, $\{1, 2\}$, $\{3, 4\}$, $\{5, 6\}$, $\{7, 8\}$, $\{9, 10\}$, and $\{11, 12\}$-th block rows, are used for the layered decoding with $p = 6$.

Figure 3 compares the simulation and analytical results for various decoding schemes. Using (5), (6), (8), and (10), the analytical BER values are calculated. In Fig. 3, A and S stand for the analytical and simulation results, respectively. We can see that the simulation results are fairly well matched with the analytical results.

Now, we confirm the tendency for the irregular B-LDPC code of length 2304 and rate $1/2$ for the practical IEEE 802.16e standards [5]. For this irregular B-LDPC code, six subsets of check nodes, $\{1, 10\}$, $\{2, 11\}$, $\{3, 5\}$, $\{4, 6\}$, $\{7, 9\}$, and $\{8, 12\}$-th block rows of 12×24 model ma-

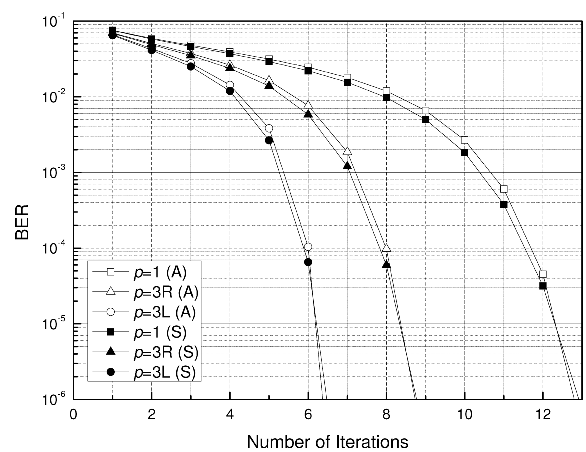
3548				1334				2990				1831				4167				2902			
	1648				2497				2598				1051				1985				1207		
		264				1110				2419				2957				3465				362	
			4087				2686				2660				3461				2455				2841
1268							2629			2205			3588				761						3806
	227			4094							1248			2637		585						2448	
		1162			1518			2310							2985				2253		2905		
			1666			2142			2422			2007						2475		1129			
1428						2103			94						240			62		1173			
	429						1763			515		4001					2891						1180
		1451		3778							3362		2529			3233						4024	
			304		2297			0						3397					1793		4055		

(a) (3,6) regular B-LDPC code.

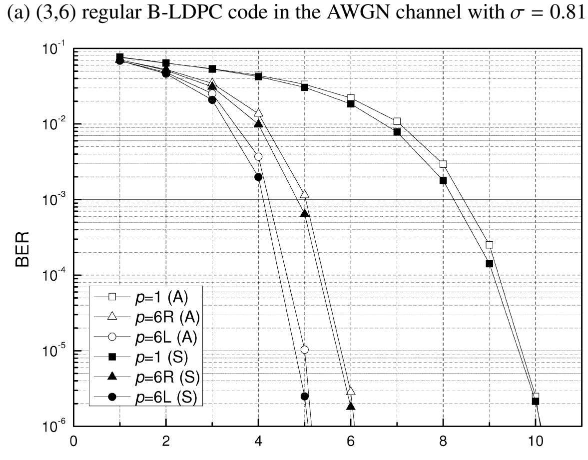
99		1447		2529				2881		3568		2205				2622		2692		2443			
	1244				2058	3731			3260				1044	3142			93				123	3340	
246		7				1377		1301		1224				4145		896		748				362	
			2117	118			3543				442	252			2548				2113	815			529
3375		2327					778	1265		3728			3299				3875			404			1742
	1432			2956		2349					3909	1221		434		4006		3520				1649	
2883		3582			3468			4165		119					1654				845		3521	2377	
			2307	2020		3402			2067			2889		2323		3978		250		2366			
2454		3289				1207			1577					2534	963	2155		1563		1269			
	4090			3840			1528	549		2198		2189					2431					1417	2857
1658		2101		1377							3414	3921	3894			1673		1411				2841	
			493		1198	715		0		3874				114					2827	3235	2836		

(b) Irregular B-LDPC code.

Fig. 2 Model matrices of B-LDPC code of rate 1/2.



(a) (3,6) regular B-LDPC code in the AWGN channel with $\sigma = 0.81$.



(b) Irregular B-LDPC code in the AWGN channel with $\sigma = 0.77$.

Fig. 3 Comparison of analytical and simulated BER performances of the layered decoding and the horizontal sequential decoding with random partitioning of B-LDPC codes of length 100248 and rate 1/2.

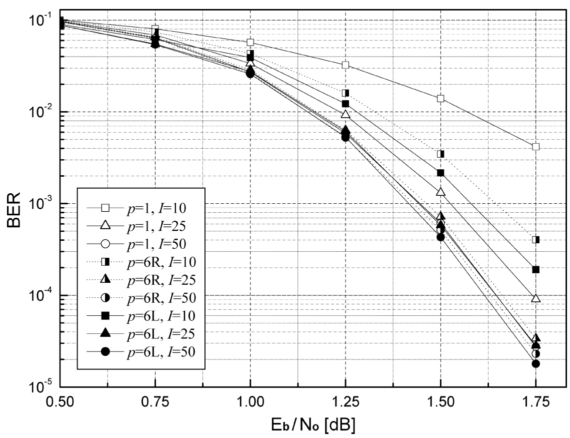


Fig. 4 BER performance of irregular B-LDPC code of length 2304 and rate 1/2 for IEEE 802.16e using various decoding schemes and iterations with $p = 6$.

trix, are used for the layered decoding with $p = 6$. Figure 4 compares the BER performance of irregular B-LDPC code using the flooding scheduling ($p = 1$), random partitioning ($p = 6$), and layered decoding ($p = 6$) for the various number I of iterations. The performance improves as p increases and the layered decoding gives fastest convergence speed as expected.

We can conclude that by using the layered decoding, the overall computational decoding complexity can be reduced because the same decoding performance can be achieved for fewer iterations. Also, LDPC codes with the structure to support the layered decoding are good to implement the decoder. Through simulation, we observed that the performance of B-LDPC codes with the dual-diagonal parity structure do not follow the analytical result. We guess that it is because the dual-diagonal parity structure of degree-2 parity nodes substantially decreases the randomness of B-LDPC code. Therefore, it will be an interesting research problem to investigate the effect of the dual-diagonal parity structure on the decoding performance.

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