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Results in Mathematics

On Eigenvalues of Row-Inverted Sylvester Hadamard Matrices

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Abstract. In this paper, the eigenvalues of row-inverted $2^n \times 2^n$ Sylvester Hadamard matrices are derived. Especially when the sign of a single row or two rows of a $2^n \times 2^n$ Sylvester Hadamard matrix are inverted, its eigenvalues are completely evaluated. As an example, we completely list all the eigenvalues of 256 different row-inverted Sylvester Hadamard matrices of size 8.

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1. Introduction

A Hadamard matrix of order N is an $N \times N$ matrix of $+1$'s and -1's such that any pair of distinct rows are orthogonal. Various construction methods including Sylvester construction, Paley construction, and Williams construction are known. The eigenvalues of Hadamard matrices are known only for Sylvester Hadamard matrices of size 2^n. It is known that Hadamard transform is an orthogonal transform with practical purpose for representing signals and images especially for the data compression [3] [5]. A complete set of 2^n Walsh functions of order n gives a Hadamard matrix H_{2^n}. Walsh–Hadamard transform (WHT) is used for the Walsh representation of the data sequences in image coding and for signature sequence in the CDMA mobile communication systems [6]. Hadamard matrices can be used to build error-correcting codes, in particular, the Reed–Muller codes [7]. In such applications, several rows of Hadamard matrices are often inverted and it is very interesting to find the eigenvalues of row-inverted Hadamard matrices.

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In this paper, the eigenvalues of some row-inverted $2^n \times 2^n$ Sylvester Hadamard matrices are derived. The paper is organized as follows: After reviewing some basic facts on Sylvester Hadamard matrices in Section 2, we derive the eigenvalues of single-row-inverted and double-rows-inverted Hadamard matrices in Section 3. Finally in Section 4, we completely find out all the eigenvalues of 256 different row-inverted Sylvester Hadamard matrices of size 8 as an example.

2. Preliminaries

Let $N = 2^n$ for a positive integer n. Let H_N be the $N \times N$ Sylvester Hadamard matrix. It is well known that Sylvester Hadamard matrices can be recursively constructed using Kronecker product as

$$H_{2N} = H_2 \otimes H_N = \begin{bmatrix} +1 & +1 \\ +1 & -1 \end{bmatrix} \otimes H_N \, .$$

From the definition above, it is not difficult to see that the $(l, m)^{th}$ entry $h_{l,m}$ of H_N can be expressed in terms of l and m as follows:

For an integer l, $0 \leq l \leq 2^n - 1$, let $\mathbf{b}(l) = (l_1, l_2, \ldots, l_n)$ be the vector in Z_2^n corresponding to the binary representation of the integer l. Also, let $S(l)$ be the set of positions $k, 1 \leq k \leq n$ such that $l_k = 1$, and $s(l)$ be the cardinality of the set $S(l)$. For example, if $n = 4$, $l = 13$ gives $\mathbf{b}(13) = (1101)$, $S(13) = \{1, 2, 4\}$, and $s(13) = 3$. Then, the $(l, m)^{th}$ entry $h_{l,m}$ of H_N is

$$h_{l,m} = (-1)^{\mathbf{b}(l)\,\mathbf{b}(m)^T} = (-1)^{|S(l) \cap S(m)|} \, . \tag{1}$$

The eigenvalues of Sylvester Hadamard matrices can be easily computed from the following well-known lemma.

Lemma 1 ([2]). *Let $C = A \otimes B$, for given two square matrices A and B. Then the eigenvalues of C are the products of those of A and B.*

If we apply Lemma 1 repeatedly, then we can see that exactly half of the eigenvalues of H_N are $+2^{n/2}$ and the other half are $-2^{n/2}$.

3. Eigenvalues of row-inverted Sylvester Hadamard matrices

In general, it is very difficult to predict how the eigenvalues of a matrix change when some rows are inverted (i.e., the signs of all the entries of those rows in a square matrix are inverted). But for a Sylvester Hadamard matrix H_N, we can observe that $N - 2K$ eigenvalues (among them, exactly half of them are $+2^{n/2}$) remain unchanged when K rows are inverted, provided that $K < N/2$. Let $N = 2^n$ and $U_N = \frac{1}{\sqrt{N}} H_N$ be the normalized Sylvester Hadamard matrix so that the eigenvalues of U_N are ± 1's. Given an index set $L = \{l_1, l_2, \ldots, l_K\}$, $0 \leq l_k \leq N-1$, define the matrix P_L as the $N \times N$ diagonal matrix whose (m, m)-entry is -1 if

$m \in L$ and $+1$ otherwise. Then the matrix $Q_{N,L}$ obtained by inverting those K rows of U_N belonging to the index set L can be expressed as

$$Q_{N,L} = P_L U_N \,.$$

Theorem 2. *Let $N = 2^n$ and $L = \{l_1, l_2, \ldots, l_K\}$. Then among the N eigenvalues of $Q_{N,L}$, each of ± 1 appears at least $\left(\frac{N}{2} - K\right)$ times, provided that $K < N/2$.*

Proof. Since U_N is an orthogonal matrix, its eigenvectors are linearly independent. In fact, one can make these eigenvectors mutually orthogonal. Also, any linear combination of eigenvectors corresponding to an eigenvalue is also an eigenvector to the same eigenvalue. Let $\overrightarrow{x}_1, \overrightarrow{x}_2, \ldots, \overrightarrow{x}_{N/2}$ be the mutually orthogonal eigenvectors of U_N corresponding to the eigenvalue $+1$. Since they are linearly independent, using elementary column operations, one can make those components belonging to L of the vectors $\overrightarrow{x}_1$ through $\overrightarrow{x}_{N/2-K}$ be zero. Thus, we have

$$P_L \overrightarrow{x}_k = \overrightarrow{x}_k, k = 1, 2, \ldots, N/2 - K \,.$$

Since vectors $\overrightarrow{x}_1$ through $\overrightarrow{x}_{N/2-K}$ are eigenvectors of U_N, we have

$$Q_{N,L} \overrightarrow{x}_k = P_L U_N \overrightarrow{x}_k = P_L \overrightarrow{x}_k = \overrightarrow{x}_k \,, \quad k = 1, 2, \ldots, \quad N/2 - K \,.$$

Thus, vectors $\overrightarrow{x}_1$ through $\overrightarrow{x}_{N/2-K}$ are also eigenvectors of $Q_{N,L}$ with eigenvalue $+1$. Similar argument can be made for the eigenvectors $\overrightarrow{y}_1, \overrightarrow{y}_2, \ldots, \overrightarrow{y}_{N/2}$ of U_N corresponding to the eigenvalue -1. Therefore, U_N and $Q_{N,L}$ share $(\frac{N}{2} - K)$ eigenvalues of $+1$ and $(\frac{N}{2} - K)$ eigenvalues of -1. □

Theorem 2 tells us that we only have to find out those $2|L|$ new eigenvalues in order for identifying all the eigenvalues of $Q_{N,L}$. The following lemma tells us about the trace of $Q_{N,L}$.

Lemma 3. *Let $N = 2^n$ and $L = \{l_1, l_2, \ldots, l_K\}$. Let p be the number of l in L such that the binary representation of l has even weight. Then,*

$$tr(Q_{N,L}) = \frac{2(K - 2p)}{\sqrt{N}} \,.$$

Proof. When the binary representation of l has even weight, $u_{l,l} = \frac{1}{\sqrt{N}}$. Since $tr(U_N) = 0$, we have

$$tr(Q_{N,L}) = p\frac{(-2)}{\sqrt{N}} + (K - p)\frac{2}{\sqrt{N}} = \frac{2(K - 2p)}{\sqrt{N}} \,.$$ □

In the next theorem, we can easily obtain the two new eigenvalues of $Q_{N,L}$ when $|L| = 1$.

Theorem 4. *Let $N = 2^n$ and $L = \{l\}$. Then the eigenvalues of $Q_{N,\{l\}}$ are ± 1 with multiplicity $(\frac{N}{2} - 1)$ each, and $\frac{(-1)^{s(l)}}{\sqrt{N}}(-1 \pm i\sqrt{N-1})$.*

Proof. From Theorem 2, we only have to identify the two new eigenvalues. Since $\det(Q_{N,\{l\}}) = -1$, the product of these two new eigenvalues must be 1. Now, due to the fact that every eigenvalue of $Q_{N,L}$, which is an orthogonal matrix has unit magnitude and that any complex eigenvalue appears in pair with its complex conjugate, we can say without loss of generality that the two new eigenvalues are $a \pm i\sqrt{1-a^2}$. From the fact that the sum of all the eigenvalues of any square matrix is its trace, we have

$$tr(Q_{N,\{l\}}) = 2a\,. \tag{2}$$

From Lemma 3, we have

$$tr(Q_{N,\{l\}}) = -2\frac{h_{l,l}}{\sqrt{N}}\,. \tag{3}$$

Thus, two new eigenvalues are

$$\frac{h_{l,l}}{\sqrt{N}}\big(-1 \pm i\sqrt{N-1}\big) = \frac{(-1)^{s(l)}}{\sqrt{N}}(-1 \pm i\sqrt{N-1})\,. \qquad \square$$

Next, let us find out the eigenvalues of $Q_{N,L}$ when $|L| = 2$. To identify the four new eigenvalues, we need an additional constraint on them. The following lemma provides us this additional constraint on new eigenvalues.

Lemma 5. *Let* $L = \{l_1, l_2, \ldots, l_K\}$*. Then,* $tr(Q^2_{N,L}) = \frac{(N-2K)^2}{N}$.

Proof. Let t_l be the l^{th} diagonal entry of $Q^2_{N,L}$. Then t_l can be expressed as

$$t_l = \sum_{j=0}^{N-1} t_{l,j} q_{j,l} = \begin{cases} -1 + \frac{2|L|}{N} & \text{if} \quad l \in L \\ 1 - \frac{2|L|}{N} & \text{otherwise} \end{cases} \tag{4}$$

because $q_{j,l} = q_{l,j}$ if both j and l are in L or neither of them are in L, and $q_{j,l} = -q_{l,j}$ if only one of j and l is in L. Summing all t_l in (4) proves the lemma. $\square$

Using Lemma 5, we can obtain the four new eigenvalues of $Q_{N,L}$ when $|L| = 2$ as in the following theorem.

Theorem 6. *Let* $N = 2^n$ *and* $L = \{l, k\}$*. Then the* $(N-4)$ *eigenvalues of* $Q_{N,L}$ *are* ± 1 *with multiplicity* $(\frac{N}{2} - 2)$ *each, and the remaining four eigenvalues are*

$$\begin{aligned} &\pm i\,, \quad \frac{-1}{\sqrt{N}}(2 \pm i\sqrt{N-4}) \quad \text{if} \quad h_{l,l} = h_{k,k} = +1 \\ &\pm i\,, \quad \frac{1}{\sqrt{N}}(2 \pm i\sqrt{N-4}) \quad \text{if} \quad h_{l,l} = h_{k,k} = -1 \\ &\pm\frac{1}{\sqrt{N}}(\sqrt{2} \pm i\sqrt{N-2}) \quad \text{if} \quad h_{l,l}h_{k,k} = -1\,. \end{aligned}$$

Proof. Again from Theorem 2, we only have to identify the four new eigenvalues. Similarly to the proof of Theorem 4, we can say without loss of generality that the four new eigenvalues are $a \pm i\sqrt{1-a^2}$ and $b \pm i\sqrt{1-b^2}$. From Lemma 3, we have

$$tr(Q_{N,\{l,k\}}) = 2(a+b) = \begin{cases} \frac{-4}{\sqrt{N}} & \text{if } h_{l,l} = h_{k,k} = +1 \\ \frac{4}{\sqrt{N}} & \text{if } h_{l,l} = h_{k,k} = -1 \\ 0 & \text{otherwise}\,. \end{cases} \tag{5}$$

Also, from Lemma 5 and the fact that λ^2 is an eigenvalue of A^2 when λ is an eigenvalue of A, we have

$$tr(Q_{N,L}^2) = \frac{(N-4)^2}{N} = N - 4 + 4(a^2 + b^2 - 1)\,. \tag{6}$$

Equation (6) is simplified as

$$a^2 + b^2 = \frac{4}{N}\,. \tag{7}$$

Finally, from (5) and (7), we have $a = 0$ and $b = \frac{-2}{\sqrt{N}}$ for $h_{l,l} = h_{k,k} = +1$, $a = 0$ and $b = \frac{2}{\sqrt{N}}$ for $h_{l,l} = h_{k,k} = -1$, and $a = \frac{\sqrt{2}}{\sqrt{N}}$ and $b = \frac{-\sqrt{2}}{\sqrt{N}}$ for $h_{l,l}h_{k,k} = -1$, which proves the theorem. □

Generally speaking, we need as many constraints as $|L|$ to identify all the eigenvalues of $Q_{N,L}$, and those constraints should be generally solvable simultaneously. Thus, it seems quite difficult to identify all the eigenvalues of $Q_{N,L}$ when $|L| \geq 3$. But, for some special types of index set L, Theorems 2, 4, and 6 can be used in combination with Lemma 1 to obtain the eigenvalues of row-inverted Sylvester Hadamard matrices. Here is one of the easiest examples. The matrix Q obtained by inverting the last 2^r rows of U_N can be expressed as

$$Q = P_{\{2^{n-r}-1\}} U_{2^{n-r}} \otimes U_{2^r}\,.$$

Thus, the eigenvalues of Q are the products of those of $P_{\{2^{n-r}-1\}} U_{2^{n-r}}$ and U_{2^r}, which are ± 1 with multiplicity $2^{n-1} - 2^r$ each, and $\pm \frac{1}{\sqrt{2^{n-r}}}(1 \pm i\sqrt{2^{n-r}-1})$ with multiplicity 2^r each.

4. Examples

In this section, we are going to identify the eigenvalues of all 255 matrices obtained from U_8 by row-inversion. Although there are $2^8 - 1$ such matrices, it is enough to consider only $2^4 - 1$ cases because of the fact that $-\lambda$ is the eigenvalue of $-A$ for every eigenvalue λ of A. Thus, we assume that $|L| \leq 4$.

Case 1: $|L| = 1$ or 2
Theorem 2 or 3 can be directly applied and the results are summarized in Table 1.
Case 2: $|L| = 3$

Table 1. Eigenvalues of $Q_{8,L}$ with $|L| = 1, 2$.

L	Eigenvalues
$\{0\}, \{3\}, \{5\}, \{6\}$	$+1, +1, +1, -1, -1, -1, -\frac{1}{\sqrt{8}}(1 \pm i\sqrt{7})$
$\{1\}, \{2\}, \{4\}, \{7\}$	$+1, +1, +1, -1, -1, -1, \frac{1}{\sqrt{8}}(1 \pm i\sqrt{7})$
$\{0,3\}, \{0,5\}, \{0,6\}, \{3,5\}, \{3,6\}, \{5,6\}$	$+1, +1, -1, -1, \pm i, -\frac{1}{\sqrt{2}}(1 \pm i)$
$\{1,2\}, \{1,4\}, \{1,7\}, \{2,4\}, \{2,7\}, \{4,7\}$	$+1, +1, -1, -1, \pm i, \frac{1}{\sqrt{2}}(1 \pm i)$
$\{0,1\}, \{0,2\}, \{0,4\}, \{0,7\}, \{3,1\}, \{3,2\}, \{3,4\}, \{3,7\}$ $\{5,1\}, \{5,2\}, \{5,4\}, \{5,7\}, \{6,1\}, \{6,2\}, \{6,4\}, \{6,7\}$	$+1, +1, -1, -1, \pm\frac{1}{2}(1 \pm i\sqrt{3})$

Inverting three rows from U_8 can be thought as inverting a single row from some matrix obtained by inverting 4 rows from U_8. Since U_8 can be expressed as

$$U_8 = U_2 \otimes U_2 \otimes U_2, \tag{8}$$

using Theorem 4 and Lemma 1, we can easily compute the eigenvalues of the matrices obtained by replacing any of U_2 in (8) by its single-row inverted versions. Moreover, all the matrices so obtained (except for U_8 itself) have the property that each eigenvalue is repeated twice. Let us define

$$A = Q_{2,\{0\}} = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 & -1 \\ +1 & -1 \end{bmatrix}$$

and $B = -A$. Then we have

$$A \otimes U_2 \otimes U_2 = Q_{8,\{0,1,2,3\}} \tag{9}$$
$$A \otimes U_2 \otimes B = Q_{8,\{0,2,5,7\}} \tag{10}$$
$$A \otimes B \otimes U_2 = Q_{8,\{0,1,6,7\}} \tag{11}$$
$$A \otimes A \otimes A = Q_{8,\{0,3,5,6\}} \tag{12}$$
$$U_2 \otimes U_2 \otimes A = Q_{8,\{0,2,4,6\}} \tag{13}$$
$$U_2 \otimes A \otimes U_2 = Q_{8,\{0,1,4,5\}} \tag{14}$$
$$U_2 \otimes A \otimes B = Q_{8,\{0,3,4,7\}} \tag{15}$$
$$B \otimes U_2 \otimes U_2 = Q_{8,\{4,5,6,7\}} \tag{16}$$
$$B \otimes U_2 \otimes B = Q_{8,\{1,3,4,6\}} \tag{17}$$
$$B \otimes B \otimes U_2 = Q_{8,\{2,3,4,5\}} \tag{18}$$
$$B \otimes A \otimes A = Q_{8,\{1,2,4,7\}} \tag{19}$$
$$U_2 \otimes U_2 \otimes B = Q_{8,\{1,3,5,7\}} \tag{20}$$
$$U_2 \otimes B \otimes U_2 = Q_{8,\{2,3,6,7\}} \tag{21}$$
$$U_2 \otimes B \otimes B = Q_{8,\{1,2,5,6\}}\,. \tag{22}$$

Using Lemma 1, we can compute the eigenvalues of each of above 14 matrices, and they are either $\{\pm 1, \pm i\}$ with multiplicity 2 each, or $\{\pm(\frac{1}{\sqrt{2}} \pm i\frac{1}{\sqrt{2}})\}$ with multiplicity 2 each. The set of 14 index sets appearing in (9) through (22) has a very interesting property that any index set L such that $|L| = 3$ is contained in some of these 14 index sets. In fact, this set of 14 4-sets is the block design such that every triplet from $Z_8 = \{0, 1, 2, \ldots, 7\}$ appears exactly once in some block, namely, a Steiner system $S(3, 4, 8)$. Note that a Steiner system $S(t, k, v)$ is a collection of distinct k-subsets (called *blocks*) of a v-set with the property that any t-subset of the v-set is contained in exactly one block. The following corollary is due to Theorem 1.

Corollary 7. *Let $J \subset L$ such that $|J| = 3$, where L is one of the above 14 index sets. Then $Q_{8,J}$ and $Q_{8,L}$ share four eigenvalues, either $\{\pm 1, \pm i\}$ or $\{\pm(\frac{1}{\sqrt{2}} \pm i\frac{1}{\sqrt{2}})\}$.*

Here, we omit the proof, because it follows exactly the same trail as the proof of Theorem 2. Now, we are ready to find out all the eigenvalues of $Q_{8,J}$. Given an index set J, $|J| = 3$, first, among the 14 index sets appearing in (9) through (22), we first identify the index set L containing J. Then the distinct four eigenvalues of $Q_{8,L}$ are also the eigenvalues of $Q_{8,J}$. Since $det(Q_{8,J}) = -1$ and this is the product of all eight eigenvalues of $Q_{8,J}$, there are two different cases to consider.

Case 2-1:
When $\pm(\frac{1}{\sqrt{2}} \pm i\frac{1}{\sqrt{2}})$ are eigenvalues of $Q_{8,J}$, then the product of the remaining four eigenvalues must be -1 and we may set the four remaining eigenvalues as ± 1 and $a \pm i\sqrt{1-a^2}$ for $a \neq \pm 1$. Using Lemma 3, we have

$$tr(Q_{8,J}) = 2a\,, \tag{23}$$

which gives us the value of a.

Case 2-2:
When $\{\pm 1, \pm i\}$ are eigenvalues of $Q_{8,J}$, then the product of the remaining four eigenvalues must be $+1$, so we may set the four remaining eigenvalues as $a \pm i\sqrt{1-a^2}$ and $b \pm i\sqrt{1-b^2}$ for $a \neq \pm 1$ and $b \neq \pm 1$. This time, we can use Lemmas 3 and 5 to obtain

$$tr(Q_{8,J}) = 2(a+b)\,, \tag{24}$$

$$tr(Q_{8,J}^2) = 4(a^2+b^2-1)\,. \tag{25}$$

Table 2 contains the complete list of eigenvalues of $Q_{8,L}$ for $|L| = 3$. Among all the $\binom{8}{3} = 56$ different J's, we show two examples, one for each case.

Example 8. When $J = \{1, 2, 3\}$, the 4-set containing J is $\{0, 1, 2, 3\}$ in (9). Thus, four eigenvalues of Q_J are $\pm(\frac{1}{\sqrt{2}} \pm i\frac{1}{\sqrt{2}})$. From Lemma 3, we have $tr(Q_J) = \frac{2}{\sqrt{8}} = 2a$. Therefore, the eigenvalues of $Q_{8,\{1,2,3\}}$ are

$$\pm\left(\frac{1}{\sqrt{2}} \pm i\frac{1}{\sqrt{2}}\right), \quad \pm 1\,, \quad \frac{1}{\sqrt{8}} \pm i\frac{\sqrt{7}}{\sqrt{8}}\,.$$

Example 9. When $J = \{2, 5, 7\}$, the 4-set containing J is $\{0, 2, 5, 7\}$ in (10). Thus, four eigenvalues of $Q_{8,J}$ are ± 1 and $\pm i$. Using Lemmas 3 and 5, (24) and (25)

TABLE 2. Eigenvalues of $Q_{8,L}$ with $|L| = 3$.

L	Eigenvalues
$\{0,1,2\}$, $\{0,1,4\}$, $\{0,2,4\}$, $\{0,3,5\}$, $\{0,3,6\}$, $\{0,5,6\}$, $\{1,2,3\}$, $\{1,3,7\}$, $\{1,4,5\}$, $\{1,5,7\}$, $\{2,3,7\}$, $\{2,4,6\}$, $\{2,6,7\}$, $\{3,5,6\}$, $\{4,5,7\}$, $\{4,6,7\}$	$\pm 1,\ \pm\frac{1}{\sqrt{2}}(1\pm i),\ \frac{1}{\sqrt{8}}(1\pm i\sqrt{7})$
$\{0,1,3\}$, $\{0,1,5\}$, $\{0,2,3\}$, $\{0,2,6\}$, $\{0,4,5\}$, $\{0,4,6\}$, $\{1,2,4\}$, $\{1,2,7\}$, $\{1,3,5\}$, $\{1,4,7\}$, $\{2,3,6\}$, $\{2,4,7\}$, $\{3,5,7\}$, $\{3,6,7\}$, $\{4,5,6\}$, $\{5,6,7\}$	$\pm 1,\ \pm\frac{1}{\sqrt{2}}(1\pm i),\ -\frac{1}{\sqrt{8}}(1\pm i\sqrt{7})$
$\{0,1,7\}$, $\{0,2,7\}$, $\{0,4,7\}$, $\{1,2,5\}$, $\{1,2,6\}$, $\{1,3,4\}$ $\{1,4,6\}$, $\{1,6,7\}$, $\{2,3,4\}$, $\{2,4,5\}$, $\{2,5,7\}$, $\{3,4,7\}$	$\pm 1, \pm i,\ \frac{1+\sqrt{17}}{2\sqrt{8}} \pm i\frac{\sqrt{14-2\sqrt{17}}}{2\sqrt{8}}$ $\frac{1-\sqrt{17}}{2\sqrt{8}} \pm i\frac{\sqrt{14+2\sqrt{17}}}{2\sqrt{8}}$
$\{0,1,6\}$, $\{0,2,5\}$, $\{0,3,4\}$, $\{0,3,7\}$, $\{0,5,7\}$, $\{0,6,7\}$ $\{1,3,6\}$, $\{1,5,6\}$, $\{2,3,5\}$, $\{2,5,6\}$, $\{3,4,5\}$, $\{3,4,6\}$	$\pm 1,\ \pm i,\ \frac{-1-\sqrt{17}}{2\sqrt{8}} \pm i\frac{\sqrt{14-2\sqrt{17}}}{2\sqrt{8}}$ $\frac{-1+\sqrt{17}}{2\sqrt{8}} \pm i\frac{\sqrt{14+2\sqrt{17}}}{2\sqrt{8}}$

TABLE 3. Eigenvalues of $Q_{8,L}$ with $|L| = 4$.

L	Eigenvalues
$\{0,1,6,7\}$, $\{0,2,5,7\}$, $\{0,3,4,7\}$	$\pm 1,\ \pm 1,\ \pm i,\ \pm i$
$\{0,1,2,3\}$, $\{0,3,5,6\}$, $\{0,1,4,5\}$, $\{0,2,4,6\}$	$\pm\frac{1}{\sqrt{2}}(1\pm i),\ \pm\frac{1}{\sqrt{2}}(1\pm i)$
$\{0,1,2,7\}$, $\{0,2,4,7\}$	$\pm\frac{1}{\sqrt{2}}(1\pm i),\ \frac{1-\sqrt{3}}{\sqrt{8}} \pm i\frac{1+\sqrt{3}}{\sqrt{8}},\ \frac{1+\sqrt{3}}{\sqrt{8}} \pm i\frac{1-\sqrt{3}}{\sqrt{8}}$
$\{0,1,3,6\}$, $\{0,2,3,5\}$, $\{0,5,6,7\}$ $\{0,2,5,6\}$, $\{0,3,4,6\}$, $\{0,3,5,7\}$	$\pm\frac{1}{\sqrt{2}}(1\pm i),\ \frac{-1+\sqrt{3}}{\sqrt{8}} \pm i\frac{1+\sqrt{3}}{\sqrt{8}},\ \frac{-1-\sqrt{3}}{\sqrt{8}} \pm i\frac{1-\sqrt{3}}{\sqrt{8}}$
$\{0,1,2,4\}$	$\frac{1}{\sqrt{2}}(1\pm i),\ \frac{1}{\sqrt{2}}(1\pm i),\ \frac{-1+\sqrt{3}}{\sqrt{8}} \pm i\frac{1+\sqrt{3}}{\sqrt{8}},\ \frac{-1-\sqrt{3}}{\sqrt{8}} \pm i\frac{1-\sqrt{3}}{\sqrt{8}}$
$\{0,1,3,5\}$, $\{0,2,3,6\}$, $\{0,4,5,6\}$	$\frac{-1}{\sqrt{2}}(1\pm i),\ \frac{-1}{\sqrt{2}}(1\pm i),\ \frac{1+\sqrt{3}}{\sqrt{8}} \pm i\frac{-1+\sqrt{3}}{\sqrt{8}},\ \frac{1-\sqrt{3}}{\sqrt{8}} \pm i\frac{1+\sqrt{3}}{\sqrt{8}}$
$\{0,1,4,7\}$	$\pm\frac{1}{\sqrt{2}}(1\pm i),\ \frac{1-\sqrt{3}}{\sqrt{8}} \pm i\frac{1+\sqrt{3}}{\sqrt{8}},\ \frac{1+\sqrt{3}}{\sqrt{8}} \pm i\frac{1-\sqrt{3}}{\sqrt{8}}$
$\{0,1,5,6\}$, $\{0,3,4,5\}$, $\{0,3,6,7\}$	$\pm\frac{1}{\sqrt{2}}(1\pm i),\ \frac{-1+\sqrt{3}}{\sqrt{8}} \pm i\frac{1+\sqrt{3}}{\sqrt{8}},\ \frac{-1-\sqrt{3}}{\sqrt{8}} \pm i\frac{1-\sqrt{3}}{\sqrt{8}}$
$\{0,1,2,5\}$, $\{0,1,2,6\}$, $\{0,1,3,4\}$, $\{0,1,3,7\}$, $\{0,1,4,6\}$, $\{0,1,5,7\}$, $\{0,2,3,4\}$, $\{0,2,3,7\}$, $\{0,2,4,5\}$, $\{0,2,6,7\}$, $\{0,4,5,7\}$, $\{0,4,6,7\}$	$\pm e^{i\frac{1}{8}\pi},\ \pm e^{i\frac{3}{8}\pi},\ \pm e^{i\frac{5}{8}\pi},\ \pm e^{i\frac{7}{8}\pi}$

become $tr(Q_{8,J}) = \frac{2}{\sqrt{8}} = 2(a+b)$ and $tr(Q_{8,J}^2) = \frac{4}{8} = 4(a^2+b^2-1)$. Therefore, the eigenvalues of $Q_{8,\{2,5,7\}}$ are

$$\pm 1, \quad \pm i, \quad \frac{1+\sqrt{17}}{2\sqrt{8}} \pm i\frac{\sqrt{14-2\sqrt{17}}}{2\sqrt{8}}, \quad \frac{1-\sqrt{17}}{2\sqrt{8}} \pm i\frac{\sqrt{14+2\sqrt{17}}}{2\sqrt{8}}.$$

We have eight more whose eigenvalues are easily obtained by using Lemma 1 and Theorem 4. They are

$$Q_{2,\{0\}} \otimes Q_{4,\{3\}} = Q_{8,\{0,1,2,7\}}\,, \qquad Q_{2,\{0\}} \otimes Q_{4,\{2\}} = Q_{8,\{0,1,3,6\}}\,,$$
$$Q_{2,\{0\}} \otimes Q_{4,\{1\}} = Q_{8,\{0,2,3,5\}}\,, \qquad Q_{2,\{1\}} \otimes Q_{4,\{0\}} = Q_{8,\{0,5,6,7\}}\,,$$
$$Q_{4,\{3\}} \otimes Q_{2,\{0\}} = Q_{8,\{0,2,4,7\}}\,, \qquad Q_{4,\{2\}} \otimes Q_{2,\{0\}} = Q_{8,\{0,2,5,6\}}\,,$$
$$Q_{4,\{1\}} \otimes Q_{2,\{0\}} = Q_{8,\{0,3,4,6\}}\,, \qquad Q_{4,\{0\}} \otimes Q_{2,\{1\}} = Q_{8,\{0,3,5,7\}}\,.$$

Case 3: $|L| = 4$
Again, due to symmetry, we only consider the 4-sets L such that $0 \in L$. There are $\binom{7}{3} = 35$ such 4-sets. We already settled down seven of them in Case 2. In the remaining 20 cases, the direct application of the theorems in this paper is not possible. But, fortunately their characteristic polynomials are easily factored so that we can evaluate the eigenvalues. The eigenvalues of $Q_{8,L}$ for the case $|L| = 4$ are listed in Table 3.

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