A New Performance Measure Using k-Set Correlation for Compressed Sensing Matrices

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***Abstract*—In this letter, a new performance measure for compressed sensing matrices is proposed. This new measure is based on the k-set correlation vectors whose components consist of the correlation values between two columns in the k-column submatrices of a sensing matrix. This measure is highly related to the restricted isometry property (RIP). And the proposed measure has less computational complexity than the condition number approach which is a typical approach for performance prediction with RIP check. It is shown by simulation that the proposed scheme works well as a performance measure for the compressed sensing matrices.**

***Index Terms*—Coherence, compressed sensing, k-set correlation, restricted isometry property (RIP), sequences.**

I. Introduction

In compressed sensing, a k-sparse message vector $\mathbf{x}$, i.e., $||\mathbf{x}||_0 \leq k$, of length N is compressed to the measurement vector $\mathbf{y}$ of length M ($M \ll N$) by an $M \times N$ sensing matrix Φ. If a "good" sensing matrix is used for proper k, M, and N, the original message vector can be recovered from the measurement vector with high probability.

It is well known that Gaussian or Bernoulli random matrices with some constraints are good candidates for sensing matrices. For this reason, random matrices have been widely used as sensing matrices. But some recent researches showed that well-designed deterministic sensing matrices have better performance and less complexity for signal reconstruction compared to random sensing matrices [1]–[3]. Therefore, design of the "good" sensing matrix becomes one of the important issues for compressed sensing. In this context, some kind of performance measure for compressed sensing matrices is required to predict the reconstruction performance of the candidate sensing matrices.

Restricted isometry property (RIP) [4] is the most well-known measure for performance of sensing matrices. If a sensing matrix Φ whose column vectors have unit norm satisfies

$$(1-\delta_k)\,||\mathbf{x}||^2 \leq ||\Phi\mathbf{x}||^2 \leq (1+\delta_k)\,||\mathbf{x}||^2 \tag{1}$$

for all possible k-sparse message vectors $\mathbf{x}$ with restricted isometry constant δ_k, then Φ is said to obey k-RIP with δ_k. The RIP with suitable constant δ_k guarantees perfect reconstruction [4], but it is very hard to check whether a sensing matrix satisfies RIP or not. The partial RIP check for performance prediction is possible in practical with condition number approach.

Coherence, the maximal correlation between two columns in a sensing matrix, is also a well-known performance measure for sensing matrices. Several measures using coherence and/or its modifications were proposed to predict the reconstruction performance of sensing matrices [1], [2], [5], [6]. However, direct performance comparison between sensing matrices is difficult with these measures.

In this letter, we introduce a new performance measure for sensing matrices using k-set correlation vectors whose components consist of correlation values between two columns in the k-column submatrix of a sensing matrix. We could predict that the sensing matrices which have lower probability of having large values of l_2-norms of k-set correlation vectors achieve better reconstruction performance. The proposed performance measure in this letter is the standard deviation of l_2-norm of k-set correlation vectors. This measure is appropriate to estimate the portion of large l_2-norm of k-set correlation vectors since the average of l_2-norm of k-set correlation vectors is nearly constant for various sensing matrices with same parameters. And the proposed measure has less computational complexity than condition number approach which is a typical approach for performance prediction with RIP check. It is also shown by simulation that the reconstruction performance of the sensing matrices can be well lined up by the proposed measure.

II. New Measure for the Performance of Compressed Sensing Matrices

Before introducing a new measure for sensing matrices, let us define the k-set correlation vector [7]. Let ϕ_j be the j-th column vector of an $M \times N$ sensing matrix Φ and $\phi_j(i)$ be the i-th component of column vector ϕ_j. The k-set correlation vector of a sensing matrix Φ is a $\binom{k}{2}$-tuple vector $\mathbf{c}$, whose component is $c_{g,h} = \sum_{i=1}^{M} \phi_g(i)\overline{\phi_h(i)}$ for $g, h \in U, g > h$, where U is a k-subset of the column index set $\{1, 2, \ldots, N\}$. Certainly, there are $\binom{N}{k}$ distinct k-set correlation vectors for an $M \times N$ sensing matrix. Let $L^{(U)}$, simply L, be the l_2-norm of k-set correlation vectors for the column index set U as

$$L^{(U)} = \sqrt{\sum_{g,h \in U, g>h} c_{g,h}^2}. \tag{2}$$

In this letter, we propose the standard deviation of l_2-norm of k-set correlation vectors as the performance measure for the

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sensing matrices. The l_2-norm of k-set correlation vector is highly related with restricted isometry constant δ_k in (1).

Let $x(i)$ be the i-th component of k-sparse message vector $\mathbf{x}$. Assume that ϕ_i's have the unit norm. Then it follows that

$$\begin{aligned}||\Phi\mathbf{x}||^2 &= \sum_{i=1}^{M}\left[\left(\sum_{g=1}^{N}\phi_g(i)x(g)\right)\overline{\left(\sum_{h=1}^{N}\phi_h(i)x(h)\right)}\right]\\ &= ||\mathbf{x}||^2 + \sum_{i=1}^{M}\sum_{g,h=1,g\neq h}^{N} x(g)\overline{x(h)}\phi_g(i)\overline{\phi_h(i)}. \quad (3)\end{aligned}$$

Note that there are at most k nonzero elements in the vector $\mathbf{x}$. Let S be the support set of $\mathbf{x}$ and $x_{g,h} = x(g)\overline{x(h)}$ for $g, h \in S$. Then from (1) and (3), we have

$$\left|\sum_{g,h\in S, g\neq h} x_{g,h}c_{g,h}\right| \leq \delta_k\, ||\mathbf{x}||^2. \quad (4)$$

The Cauchy–Schwartz inequality says that

$$\left|\sum_{\substack{g,h\in S,\\ g\neq h}} x_{g,h}c_{g,h}\right| \leq \sqrt{\left(\sum_{\substack{g,h\in S,\\ g\neq h}}|x_{g,h}|^2\right)\left(\sum_{\substack{g,h\in S,\\ g\neq h}}|c_{g,h}|^2\right)}. \quad (5)$$

It is easy to derive that

$$\begin{aligned}\sum_{g,h\in S,g\neq h}|x_{g,h}|^2 &= \sum_{g\in S}|x(g)|^2\sum_{h\in S}\left|\overline{x(h)}\right|^2 - \sum_{g\in S}|x(g)|^4\\ &= ||\mathbf{x}||^4 - \frac{1}{k}\left(\sum_{h\in S}1^2\sum_{g\in S}|x(g)|^4\right) \quad (6)\\ &\leq \frac{k-1}{k}||\mathbf{x}||^4\end{aligned}$$

where Cauchy–Schwartz inequality is applied to the second term of the right-hand side of (6). Then from (5), we have

$$\left|\sum_{g,h\in S,g\neq h} x_{g,h}c_{g,h}\right| \leq \sqrt{\frac{k-1}{k}\sum_{g,h\in S,g\neq h}|c_{g,h}|^2}\cdot||\mathbf{x}||^2. \quad (7)$$

Let $\delta(S)$ be the minimum restricted isometry constant which satisfies (1) for all message vectors $\mathbf{x}$ whose support sets are subsets of S. Then $\delta(S)$ is the minimum value that guarantees (4) for all $\mathbf{x}$ with the given S. From this and (7), we have

$$\delta(S) \leq \sqrt{\frac{k-1}{k}\sum_{g,h\in S,g\neq h}|c_{g,h}|^2} = \sqrt{\frac{2(k-1)}{k}}L^{(S)}. \quad (8)$$

In other words, $\delta(S)$ is upper bounded by constant multiple of the l_2-norm of k-set correlation vectors in S. If $L^{(S)} < T$ for all support sets S with $|S| = k$, then Φ satisfies k-RIP with δ_k which is less than or equal to $\sqrt{2(k-1)T/k}$.

Reducing the maximal l_2-norm of k-set correlation vectors enough to ensure RIP is a too restrictive requirement for the sensing matrix design, in the sense that RIP is not a necessary condition if we only want the signal reconstruction to be successful with high probability, although not always. Thus we want to loosen the RIP condition by an average-sense measure. According to (8), if a sensing matrix has low probability of having large values in l_2-norm L of k-set correlation vectors, the value of $\delta(S)$ becomes small with high probability, that is, k-column submatrices of the sensing matrix satisfy RIP with high probability. Thus, we could predict that sensing matrices which have lower probability of having large values in L achieve better reconstruction performance.

At this point, we need some kind of quantitative measure other than "comparing the distribution" for the performance comparison of sensing matrices. Therefore, we propose the standard deviation of l_2-norm of k-set correlation vectors as a performance measure for compressed sensing matrices. It will be shown in the next section that the average of L is almost constant for various sensing matrices for fixed M, N, k. Thus, the standard deviation of L could estimate the portion of large values in L and it will predict the reconstruction performance of sensing matrices well. Furthermore, computation of standard deviation of L can be estimated by computing not all $\binom{N}{k}$ k-set correlation vectors but part of them.

III. Analysis of the Proposed Measure

In this section, we will discuss the properness of the proposed measure and comparison with the condition number approach for the RIP.

A. Consistency of the Average of l_2-Norm

As mentioned in Section II, the sensing matrix having large values in L with lower probability will have better reconstruction performance. Thus, in order to use standard deviation of L as a performance measure, we have to ensure that the average of L is nearly constant for various sensing matrices for fixed M, N, k.

It is easy to derive from (2) that $E[L^2] = \binom{k}{2}\cdot E[c^2]$, where c is a correlation value between two columns in k-column submatrix. And $E[L] = \sqrt{E[L^2] - \mathrm{var}[L]} \approx \sqrt{E[L^2]}$ for large k since $E[L^2]$ is increasing but $\mathrm{var}[L]$ is almost invariant (will be shown in Section III-B) while k is increasing. From the proof of Welch bound, $E[c^2] \geq (N-M)/M(N-1)$ [8]. Assume that well-designed deterministic sensing matrices have nearly optimal correlation property in average sense, i.e., $E[c^2] \approx (N-M)/M(N-1)$. Then we have

$$E[L] \approx \sqrt{\binom{k}{2}\cdot E[c^2]} \approx \sqrt{\frac{(N-M)k(k-1)}{2M(N-1)}}. \quad (9)$$

From (9), we can confirm that averages of L of well-designed deterministic sensing matrices are almost constant for the given M, N, k. And for $N \gg M$, it is easy to show that $E[L] \lessapprox \sqrt{k(k-1)/2M}$. This value is very similar with the average $\sqrt{(2k(k-1)-1)/4M}$ of L of random matrices which is estimated by chi-square approximation in Section III-B. That is, the average of L of random sensing matrices and deterministic sensing matrices with near-optimal correlation property are nearly constant.

B. Consistency of the Standard Deviation of l_2-Norm

In this subsection, we argue that the proposed measure is almost constant for the given sensing matrix Φ and varying k. We prove this for the random sensing matrices with constant magnitude of components by showing that the distribution of L for the random sensing matrices is approximately chi-square distributed. We cannot prove this for general sensing matrices, but we verify that by the numerical analysis.

First, we will prove that the distribution of L is approximately chi-square distributed for random sensing matrices. Assume that Φ is an $M \times N$ random sensing matrix, whose components are complex numbers with magnitude $1/\sqrt{M}$ and uniform phase in $[0, 2\pi)$. The components of the k-set correlation vector are $c_{g,h} = \sum_{i=1}^{M} \phi_g(i) \cdot \overline{\phi_h(i)}$ for $g, h \in U$, $g > h$, where U is a k-subset of column indices of Φ. Let $a_{g,h}$ and $b_{g,h}$ be the real and imaginary parts of $c_{g,h}$, respectively. Then L will be represented as $L = \sqrt{\sum_{g,h \in U, g>h} (a_{g,h}^2 + b_{g,h}^2)}$.

If $a_{g,h}$ and $b_{g',h'}$ are independent Gaussian r.v.'s for $g, g', h, h' \in U$, L will be chi-square distributed with $k(k-1)$ degree of freedom. But unfortunately, we partially prove the properties of $c_{g,h}$'s as in the following theorem.

Theorem 1: For sufficiently large M, $a_{g,h}$ and $b_{g',h'}$ in the above definition are Gaussian r.v.'s and $c_{g,h}$ are pairwise independent for all $g, g', h, h' \in U$, $g > h$, $g' > h'$. In addition, $a_{g,h}$ and $b_{g,h}$ are uncorrelated for $g, h \in U$, $g > h$.

Proof:

i) Gaussian: It is manifest from the central limit theorem.
ii) Pairwise independent: It is also easy to prove it.
iii) Uncorrelated: $a_{g,h}$ and $b_{g,h}$ can be rewritten as $a_{g,h} = \sum_{i=1}^{M} \cos\theta_i / M$ and $b_{g,h} = \sum_{i=1}^{M} \sin\theta_i / M$, where θ_i's are independent r.v.'s with uniform distribution in $[0, 2\pi)$. Then $\mathrm{cov}(a_{g,h}, b_{g,h}) = E\left[\sum_{i=1}^{M} \cos\theta_i \cdot \sin\theta_i / M^2\right] = 0$.

■

The above theorem is not enough to prove that L follows chi-square distribution because $a_{g,h}$ and $b_{g',h'}$ are not mutually independent. However, $a_{g,h}$ and $b_{g',h'}$ partially satisfy the independent conditions for chi-square approximation of L. Thus L can be approximated to chi-square distribution, which is confirmed by numerical analysis in Section IV-B.

It is well known that chi-square distribution with degree of freedom d could be approximated to $\mathcal{N}(\sqrt{d - 1/2}, 1/2)$. After normalization, the distribution of L is approximated to $\mathcal{N}(\sqrt{(2k(k-1) - 1)/4M},\ 1/2M)$. Thus, standard deviation of L of random matrices is not altered even if k is varying.

It is difficult to obtain the distribution of L of the sensing matrices except the random matrices. However, it is confirmed by numerical analysis in Section IV-B that standard deviation of L is almost constant with different k's. Thus we can compute the standard deviation of L for the sensing matrices with relatively small computational complexity by using small k.

C. Comparison With the Condition Number Approach

As mentioned in Section II, the performance comparison of sensing matrices using the proposed measure can be performed with only partial computation of l_2-norm of k-set correlation vectors. However, similar statistical approaches with RIP check are also possible representatively the condition number approach. In this subsection, we compare our approach with condition number approach.

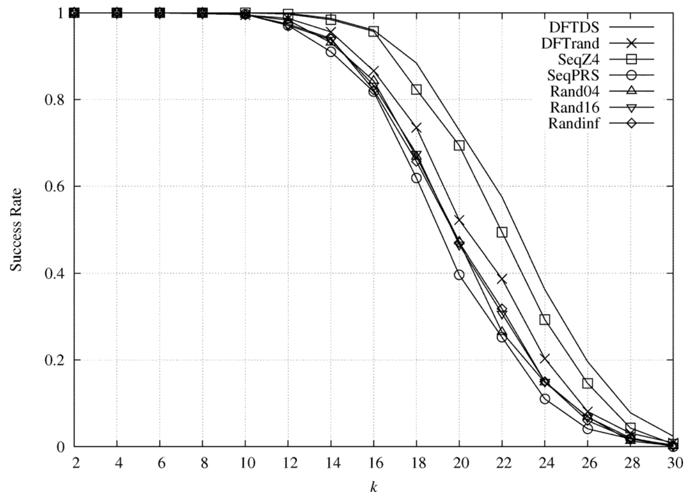

Fig. 1. Success rate of reconstruction with OMP; 65×4161 DFT matrices, 61×3844 PRS matrix, 63×4096 random and family A matrices.

First, we will introduce the condition number approach briefly. Consider $k \times k$ correlation matrix $R^{(S)}$ whose components consist of correlations between columns in k-column submatrix of the sensing matrix Φ with column indices in S, where S is a k-subset of $\{1, 2, \ldots, N\}$. Let $\kappa(R^{(S)})$ be the condition number, the ratio of the minimum eigenvalue to the maximum eigenvalue of $R^{(S)}$. If $\kappa(R^{(S)}) \le (1 + \delta_k)/(1 - \delta_k)$ for all index set S with $|S| = k$, Φ obeys k-RIP with δ_k. It is also easy to predict that if condition number $\kappa(R^{(S)})$ has large value with lower probability, the sensing matrix will have better reconstruction performance.

However, the computation of eigenvalues of a matrix requires high computational complexity. Furthermore, as mentioned in Section III-B, the proposed measure is almost constant for varying k, that is, computing the proposed measure only for small k is enough to predict the performance of sensing matrices. According to above discussion, the approach using the proposed measure has less computational complexity than condition number approach.

IV. Numerical Analysis

A. Design of Sensing Matrices

Recently, several methods for designing deterministic sensing matrices are proposed including matrix design from codes and sequences [1]–[3]. Sequences are suitable for being used as columns of the sensing matrices due to their good correlation properties. When we want to design sensing matrices from sequences, the family size of the sequences also has to be considered since large family size of sequences guarantees large N for fixed M. In this letter, we design a deterministic compressed sensing matrix using quaternary sequences of period M, called family A, defined on Z_4 [9], which have not only optimal correlation property but also large family size $M + 2$. A sensing matrix using quaternary power residue sequences [10] is also constructed. The columns of the sensing matrices constructed by the sequences consist of all shifted version of every sequence with alphabet $\{+1, -1, +j, -j\}$ in the sequence family and an all-one column.

TABLE I
STANDARD DEVIATION OF l_2-NORM OF k-SET CORRELATION VECTORS

Type	k=3	k=4	k=8	k=12	k=20
DFTDS	0	0	0	0	0
DFTrand	5.990E-2	6.045E-2	6.078E-2	6.079E-2	6.072E-2
SeqZ4	1.416E-2	1.379E-2	1.357E-2	1.353E-2	1.348E-2
SeqPRS	6.810E-2	6.692E-2	6.552E-2	6.527E-2	6.506E-2
Rand04	6.115E-2	6.178E-2	6.226E-2	6.235E-2	6.242E-2
Rand16	6.123E-2	6.181E-2	6.226E-2	6.235E-2	6.241E-2
Randinf	6.122E-2	6.180E-2	6.226E-2	6.233E-2	6.240E-2

TABLE II
AVERAGE OF l_2-NORM OF k-SET CORRELATION VECTORS

Type	k=3	k=4	k=8	k=12	k=20
DFTDS	0.213	0.302	0.651	1.000	1.696
DFTrand	0.204	0.295	0.648	0.998	1.695
SeqZ4	0.216	0.306	0.662	1.016	1.723
SeqPRS	0.209	0.304	0.669	1.030	1.750
Rand04	0.210	0.302	0.664	1.022	1.735
Rand16	0.209	0.302	0.664	1.022	1.736
Randinf	0.209	0.302	0.664	1.022	1.735

Random sensing matrices with various alphabet sizes and partial Fourier sensing matrices are also constructed for performance comparison, where each component has constant magnitude. M rows of partial Fourier matrices are selected at random or using difference set [11] from $N \times N$ discrete Fourier transform (DFT) matrix.

B. Numerical Results

Fig. 1 shows the rate of successful reconstruction per 1000 frames for the sensing matrices mentioned in Section IV-A. The matrices used for simulation are DFT sensing matrices with random or difference set row selection, random sensing matrices with several alphabet size, matrices constructed by power residue sequences and quaternary family A sequences. Size of the sensing matrices is 65×4161 for DFT matrices, 61×3844 for PRS matrix, and 63×4096 for others. Singer difference set with parameters (4161, 65, 1) is used for row selection of DFTDS matrix. Message vectors are k-sparse and their nonzero components are random complex numbers whose magnitudes are uniform in (0, 100) and phases are uniform in $[0, 2\pi)$. Orthogonal matching pursuit (OMP) [12] is used for reconstruction of the k-sparse signals. The components of the $M \times N$ sensing matrices are complex numbers with magnitude $1/\sqrt{M}$, so that each column has unit norm.

Tables I and II show the standard deviation and average of l_2-norm of k-set correlation vectors, respectively, which is simulated for 10^7 correlation vectors. From Section II, it is easily expected that the sensing matrix whose l_2-norms of k-set correlation vectors have large values with lower probability will show better reconstruction performance. Thus we could estimate from Table I that the sensing matrices in order of best to worst reconstruction performance are DFTDS, SeqZ4, DFTrand, Rand, and SeqPRS. Those are exactly matched with the observation in Fig. 1.

The consistency of the standard deviation of L with varying k mentioned in Section III-B is confirmed by Table I. And the consistency of the average of L for random matrices and deterministic sensing matrices with near-optimal correlation property mentioned in Section III-A is also confirmed by Table II. The error of the approximation seems to be negligible considering the effects of the difference of matrix size in Table II. It is also shown in Fig. 2 that L of various random sensing matrices can be approximated to the chi-square distribution. We also confirm by simulation that the proposed measure works well for different size of sensing matrices, 127×16384 and 255×65536.

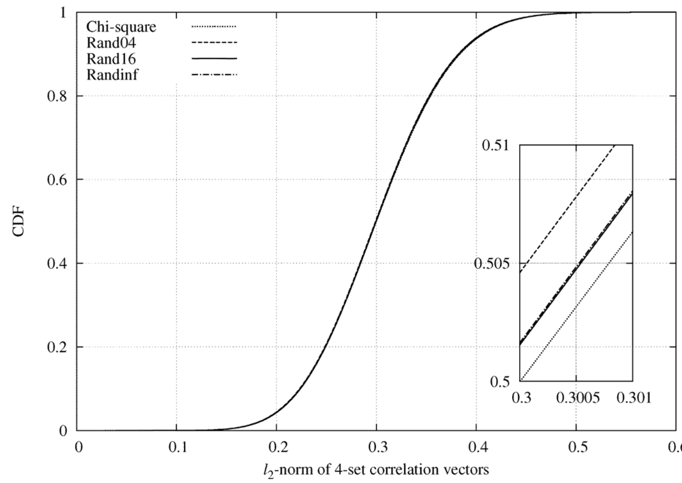

Fig. 2. Comparison between cumulative distribution of l_2-norm of 4-set correlation vectors for random sensing matrices with several alphabet size and chi-square distribution with 12 degree of freedom.

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