New Decoding Scheme for LDPC Codes Based on Simple Product Code Structure

Beomkyu Shin, Seokbeom Hong, Hosung Park, Jong-Seon No, and Dong-Joon Shin

Abstract: **In this paper, a new decoding scheme is proposed to improve the error correcting performance of low-density parity-check (LDPC) codes in high signal-to-noise ratio (SNR) region by using post-processing. It behaves as follows: First, a conventional LDPC decoding is applied to received LDPC codewords one by one. Then, we count the number of word errors in a predetermined number of decoded codewords. If there is no word error, nothing needs to be done and we can move to the next group of codewords with no delay. Otherwise, we perform a proper post-processing which produces a new soft-valued codeword (this will be fully explained in the main body of this paper) and then apply the conventional LDPC decoding to it again to recover the unsuccessfully decoded codewords. For the proposed decoding scheme, we adopt a simple product code structure which contains LDPC codes and simple algebraic codes as its horizontal and vertical codes, respectively.**

The decoding capability of the proposed decoding scheme is defined and analyzed using the parity-check matrices of vertical codes and, especially, the *combined-decodability* **is derived for the case of single parity-check (SPC) codes and Hamming codes used as vertical codes. It is also shown that the proposed decoding scheme achieves much better error correcting capability in high SNR region with little additional decoding complexity, compared with the conventional LDPC decoding scheme.**

Index Terms: **Combined-decodability, decoding, low-density parity-check (LDPC) codes, post-processing, product codes.**

I. INTRODUCTION

RECENTLY, extremely low error probability has been required in many application areas such as wireless communication systems without feedback and data storage systems. Low-density parity-check (LDPC) codes [1], [2] have become one of promising error correcting codes in these areas due to their near capacity-approaching performance. However, since finite-length LDPC codes show an error-floor problem [3] in high signal-to-noise ratio (SNR) region, it may be difficult to achieve an extremely good error correcting performance.

Many researches on the iterative decoding of LDPC codes (e.g., [4]–[16]) have been made to achieve good error correcting performance and most of them focus on improving only the decoding algorithm not changing the whole code structure. As the requirements for the error correcting performance and the complexity of decoder are getting more strict, more powerful decoding schemes for LDPC codes are needed, which could come together with some modifications of the code structure.

There are some researches focusing on post-processing after normal LDPC decoding instead of modifying the decoding algorithm itself to improve the error correcting performance of the decoder. In [13], the structure of the absorbing set is exploited by biasing the reliabilities of selected messages in post-processing. In [14], three classes of decoders are proposed to mitigate the bad effect of trapping sets [3] based on post-processing approach. In [15] and [16], the proposed decoding schemes are able to lower error floors by first observing the oscillatory behavior of the normal LDPC decoding and then applying proper post-processing schemes based on the statistics of the oscillation.

In this paper, we propose a decoding scheme for LDPC codes by using post-processing and a conventional LDPC decoder to improve the error correcting performance together with error floor. The conventional LDPC decoding is first applied to received LDPC codewords from the channel in order. After a predetermined number of codewords are decoded, we check how many word errors occurred. If there is no word error, nothing needs to be done for this group of codewords and we can move to the next group with no delay. Otherwise, we perform a proper post-processing which produces a new soft-valued codeword by combining two independently received soft-decision data for an unsuccessfully decoded codeword, and then apply the conventional LDPC decoding to it again to recover the unsuccessfully decoded codeword. For this decoding scheme with post-processing, a simple product code structure is adopted and it contains LDPC codes and simple algebraic codes as its horizontal and vertical codes, respectively.

In the encoding procedure, horizontal LDPC codewords are generated from the message and at last a few horizontal LDPC codewords are additionally generated from the previously generated LDPC codewords by using the vertical code according to the product encoding. Stacking up all these LDPC codewords will result in a codeword matrix like product code structure. In the decoding procedure, the LDPC codewords containing the message are first decoded in a row by using the conventional LDPC decoding algorithm. If some codewords are not successfully decoded, the additionally generated LDPC codewords are used to help correctly re-decode the LDPC codewords contain-

Manuscript received October 8, 2014; approved for publication by Emanuele Viterbo, Division I Editor, March 25, 2015

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Digital object identifier 10.1109/JCN.2015.000066

ing the message, where the combination of two received soft data plays a crucial role here. The details of this decoding procedure will be shown in Section II.

Also, we define and analyze the decoding capability of the proposed decoding scheme using the parity-check matrices of vertical codes and derive the *combined-decodability* for the case of single parity-check (SPC) codes and Hamming codes which are used as vertical codes. From simulations, it is shown that the proposed decoding scheme has better error correcting capability than the convectional LDPC decoding, especially, in high SNR region.

It is worth noting that the goal of this paper is to successfully decode LDPC codewords in serial order as soon as possible like the normal LDPC code transmission. We adopted the product code structure to just help the decoding of LDPC codewords when they are not successfully decoded. In this sense, the proposed decoding scheme had better not be viewed as an instance of product decoding scheme. Actually, our proposed scheme is more on a kind of post-processing scheme for LDPC codes. Compared to the normal LDPC code transmission, the proposed scheme incurs a decoding delay, additional hardware and computation complexity but it can achieve very low error rate instead. Moreover, the additional computational complexity actually becomes negligible as the SNR is getting higher because word error events seldom occur. Compared to the conventional product decoding using turbo iterations, the proposed scheme will definitely have worse error correcting performance but it requires far less memory, hardware, computation, and delay. Hence, our proposed scheme fits better than the conventional product decoding for applications which cannot have enough hardware and computational power but require short decoding delay.

Our proposed scheme can be extended to general scenarios. Any linear codes can be used as vertical codes in the proposed scheme. However, since vertical codes are only used to help decoding of LDPC codes in the proposed scheme, short codes with high rate are preferable as vertical codes. Thus, simple algebraic codes such as SPC codes or Hamming codes are appropriate for the vertical codes. Even though the vertical codes are assumed to be systematic in this paper, they do not have to be systematic for the proposed decoding scheme. The proposed decoding scheme can be applied to any other linear codes with soft-decision decoding than LDPC codes. However, undetected errors in the horizontal codewords cause a serious problem for the proposed scheme. If undetected errors seldom occur and soft-decision decoding is used for the horizontal codes such as LDPC and turbo codes [17], the proposed decoding scheme can be effectively applied.

II. A NEW DECODING SCHEME BASED ON SIMPLE PRODUCT CODE STRUCTURE

In this section, we propose a new decoding scheme for LDPC codes using the concept of product codes. In the proposed scheme, LDPC codes act as horizontal codes of the product codes and simple codes such as SPC codes or Hamming codes are used for vertical codes of the product code to help the decoding of the horizontal codes.

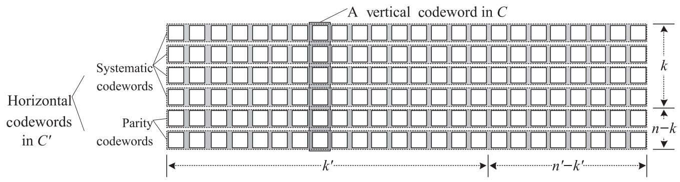


Fig. 1. Structure of a codeword matrix of product code.

A. Notations and Basic Operations

Product codes are serially concatenated codes [18], which were introduced in [19]. By using the concept of product codes, a long block code can be easily constructed by using two or more short block codes. Now, we consider two systematic linear block codes, $\mathcal{C}$ with parameters $(n, k, d_{\min})$ and $\mathcal{C}'$ with parameters $(n', k', d'_{\min})$, where n(or n'), k(or k'), and $d_{\min}$(or $d'_{\min}$) denote the code length, the number of information bits, and the minimum Hamming distance, respectively. The product code $\mathcal{P} = \mathcal{C} \otimes \mathcal{C}'$ is obtained by the following steps as illustrated in Fig. 1:

1) Placing $k \times k'$ information bits in an array of k rows and k' columns;
2) Encoding each of k rows using code $\mathcal{C}'$;
3) Encoding each of n' columns using code $\mathcal{C}$.

The codes $\mathcal{C}$ and $\mathcal{C}'$ are also called as vertical code and horizontal code of the product code $\mathcal{P}$, respectively. The parameters of the product code $\mathcal{P}$ are $(n \times n', k \times k', d_{\min} \times d'_{\min})$ and the code rate is given as $R \times R'$, where R and R' are the code rates of $\mathcal{C}$ and $\mathcal{C}'$, respectively.

In this paper, for simplicity, only binary linear codes are considered but the proposed decoding scheme can be directly applied to nonbinary codes. Let e, $0 \leqq e \leqq n$, be the number of unsuccessfully decoded horizontal codewords in a codeword matrix after the first conventional decoding of each n horizontal codewords. The parity-check matrix of a vertical code and its modifications are defined as:

- H: $m \times n$ parity-check matrix of a vertical code in the product code;
- H_E: $M \times n$ extended parity-check matrix whose rows are all possible nonzero linear combinations of rows in H, where $M = 2^m - 1$;
- H_P: $M \times e$ punctured parity-check matrix constructed by selecting e columns from H_E.

It can be assumed that H contains no all-zero column. It is clear that the length n of the vertical code also denotes the number of horizontal codewords in a codeword matrix of the product code. The column indices of H_P are same as the row indices of the unsuccessfully decoded horizontal codewords in the codeword matrix.

Let $\mathbf{c}_i = (c_{i1}, c_{i2}, \cdots, c_{in'})$, $1 \leqq i \leqq n$, denote n binary horizontal codewords and $\mathbf{r}_i = (r_{i1}, r_{i2}, \cdots, r_{in'})$ denote the *soft-decision vector* for $\mathbf{c}_i$, where r_{ij} is the log likelihood ratio (LLR) for c_{ij} calculated from the corresponding received data a_{ij} defined by

$$r_{ij} = \log\left(\frac{\Pr(a_{ij}|c_{ij} = +1)}{\Pr(a_{ij}|c_{ij} = -1)}\right).$$

Then, the check equations (or rows) in $H_E = [h_{ji}]$ describe all possible relations among $\mathbf{c}_i$'s, that is, $\sum_{i=1}^{n} h_{ji}\mathbf{c}_i = \mathbf{0}, 1 \leqq j \leqq M$. Note that these check equations (or rows) of H_E are not linearly independent. Since the vertical code is a systematic code, we can use $\mathbf{s}_i = (s_{i1}, s_{i2}, \cdots, s_{in'})$, $1 \leqq i \leqq k$, and $\mathbf{p}_i = (p_{i1}, p_{i2}, \cdots, p_{in'})$, $1 \leqq i \leqq n-k$, to denote the *systematic (horizontal) codewords* including the systematic part of the vertical code and the *parity (horizontal) codewords* including the parity part of the vertical code as given in Fig. 1, respectively.

Special algebra for LLR values in [20] can be used with small modification. We use the operation $\boxplus$ for combining two LLR values defined as

$$r_{ij} \boxplus r_{i'j} = 2\tanh^{-1}\left(\tanh\left(\frac{r_{ij}}{2}\right)\tanh\left(\frac{r_{i'j}}{2}\right)\right). \quad (1)$$

Let $\bar{\mathbf{r}}_i = (\bar{r}_{i1}, \bar{r}_{i2}, \cdots, \bar{r}_{in'})$, $1 \leqq i \leqq n$, denote the vector of LLR values computed from the horizontal codeword $\mathbf{c}_i$ itself, i.e., $\bar{r}_{ij} = -\infty$ if $c_{ij} = 1$ and $\bar{r}_{ij} = +\infty$ if $c_{ij} = 0$. Then it is easy to show that $r_{ij} \boxplus \bar{r}_{i'j}$ can be calculated using the sign change operation as

$$\begin{cases} r_{ij}, & \text{if } \bar{r}_{i'j} = +\infty; \\ -r_{ij}, & \text{if } \bar{r}_{i'j} = -\infty. \end{cases} \quad (2)$$

B. A New Decoding Scheme

In the proposed decoding scheme, the received codewords of the LDPC codes are stacked as horizontal codewords for the product code. The number of codewords in a stack is determined by the vertical codes which are used to help decoding of the horizontal codes. Next, we attempt to decode each horizontal codeword in the codeword matrix. If all the systematic codewords $\mathbf{s}_i$, $1 \leqq i \leqq k$, are successfully decoded, the decoding process will be successfully finished. Otherwise, the horizontal codewords are re-decoded using H_P of the vertical codes. The details of this procedure are explained below.

Suppose that there are e unsuccessfully decoded horizontal codewords in a codeword matrix and the jth row of H_P of the vertical codes has the minimum nonzero Hamming weight $e_{\min}$ among all rows of H_P. Clearly, it implies that $e_{\min}$ unsuccessfully decoded horizontal codewords corresponding to the nonzero elements are involved in the jth check equation of H_E of the vertical code, which is the check equation affected by the minimum number of unsuccessfully decoded horizontal codewords. Therefore, H_P is used to decide which check equation of H_E is most easily solvable, i.e., which check equation contains the fewest unsuccessfully decoded horizontal codewords. In the proposed decoding scheme, the check equation which contains the fewest unsuccessfully decoded horizontal codewords is first utilized for re-decoding.

According to the value of $e_{\min}$, the proposed decoding scheme is performed by considering the following three cases.

- Case 1) $e_{\min} = 1$:
  If there is only one unsuccessfully decoded horizontal codeword participating in a check equation of H_E, say the jth row of H_E, of the vertical code, we can correct it simply by the modulo 2 addition (denoted by $\oplus$) of the other successfully decoded horizontal codewords participating in the same check equation. Let $\mathbf{c}_i$ be the unsuccessfully decoded horizontal codeword participating in the jth check equation of H_E of the vertical code. Then we can recover $\mathbf{c}_i$ by

$$\mathbf{c}_i = \bigoplus_{\forall l \in H_E(j)\setminus\{i\}} \mathbf{c}_l \quad (3)$$

  where $H_E(j)$ is the set of indices of the nonzero elements in the jth row of H_E. As $\mathbf{c}_i$ is successfully decoded, the corresponding column of H_P is removed and the number of columns in H_P reduces by one. This process is repeatedly applied to the check equations which satisfy the $e_{\min} = 1$ condition for updated H_P.
- Case 2) $e_{\min} = 2$:
  If there are only two unsuccessfully decoded horizontal codewords participating in a check equation (i.e., a row) of H_E of the vertical code, we can re-decode the combined vector of two independently received soft-decision vectors for the same horizontal codeword similar to the type-I hybrid-automatic repeat request (H-ARQ) scheme [21]. We already know that among horizontal codewords participating in a check equation of H_E, any codeword is equivalent to the sum of all the other codewords. Thus we can derive two independently received soft-decision vectors for any of two unsuccessfully decoded horizontal codewords; (a) one from an unsuccessfully decoded horizontal codeword, (b) the other one from the soft-decision vector of the other unsuccessfully decoded horizontal codeword by changing signs of its elements according to the successfully decoded horizontal codewords participating in the same check equation.

  Let $\mathbf{c}_{i_1}$ and $\mathbf{c}_{i_2}$ be the unsuccessfully decoded horizontal codewords participating in the jth check equation of H_E of the vertical code. Then, from (3), we can have

$$\mathbf{c}_{i_1} = \mathbf{c}_{i_2} \oplus \left(\bigoplus_{\forall l \in H_E(j)\setminus\{i_1, i_2\}} \mathbf{c}_l\right). \quad (4)$$

  In addition to the soft-decision vector $\mathbf{r}_{i_1}$ for $\mathbf{c}_{i_1}$, we can derive the other soft-decision vector for $\mathbf{c}_{i_1}$ by using the relation (4) as

$$\mathbf{r}_{i_2} \boxplus \left(\boxplus_{\forall l \in H_E(j)\setminus\{i_1, i_2\}} \bar{\mathbf{r}}_l\right). \quad (5)$$

  Since it can be regarded that the horizontal codeword $\mathbf{c}_{i_1}$ is independently received twice through the channel, we can just add two soft-decision vectors as

$$\mathbf{r}_{i_1} + \left[\mathbf{r}_{i_2} \boxplus \left(\boxplus_{\forall l \in H_E(j)\setminus\{i_1, i_2\}} \bar{\mathbf{r}}_l\right)\right] \quad (6)$$

  and re-decode it by achieving approximately 3 dB performance gain similar to type-I H-ARQ scheme.

  If the re-decoding is successful, then the jth check equation of H_E contains only one unsuccessfully decoded horizontal codeword $\mathbf{c}_{i_2}$ and it can be recovered by the method in Case 1).
- Case 3) $e_{\min} \geqq 3$:

If there are three or more unsuccessfully decoded horizontal codewords participating in a check equation of H_E of the vertical code, combining the soft-decision vectors for further re-decoding is still possible. First of all, the unsuccessfully decoded horizontal codewords participating in the same check equation of H_E are partitioned into two groups and a soft-decision vector for each group is obtained by summing all the unsuccessfully decoded soft-decision vectors in that group using the operation (1). Then, for these two soft-decision vectors, the same method for two unsuccessfully decoded horizontal codewords in Case 2) is applied. We can try this process to all possible partitioning of unsuccessfully decoded horizontal codewords until the decoding succeeds and then reduces the number of unsuccessfully decoded codewords in the same check equation. However, it should be noted that more combined horizontal codewords result in less performance gain.

Let $\mathbf{c}_{i_1}, \cdots, \mathbf{c}_{i_g}, \cdots, \mathbf{c}_{i_h}, \cdots, \mathbf{c}_{i_{e_{\min}}}$ be the $e_{\min}$ unsuccessfully decoded horizontal codewords participating in the jth check equation of H_E of the vertical code. Then (3) can be rewritten as

$$\left(\bigoplus_{\forall l \in \{i_{h+1}, \cdots, i_{e_{\min}}\}} \mathbf{c}_l\right) = \left(\bigoplus_{\forall l \in \{i_1, \cdots, i_h\}} \mathbf{c}_l\right) \oplus \left(\bigoplus_{\forall l \in H_E(j) \backslash \{i_1, \cdots, i_{e_{\min}}\}} \mathbf{c}_l\right). \quad (7)$$

Two soft-decision vectors for $\bigoplus_{\forall l \in \{i_{h+1}, \cdots, i_{e_{\min}}\}} \mathbf{c}_l$ can be obtained as

$$\boxplus_{\forall l \in \{i_{h+1}, \cdots, i_{e_{\min}}\}} \mathbf{r}_l \quad (8)$$

and

$$\left(\boxplus_{\forall l \in \{i_1, \cdots, i_h\}} \mathbf{r}_l\right) \boxplus \left(\boxplus_{\forall l \in H_E(j) \backslash \{i_1, \cdots, i_{e_{\min}}\}} \bar{\mathbf{r}}_l\right). \quad (9)$$

Since it can be regarded that one horizontal codeword is independently received twice through the channel, we can add two soft-decision vectors in (8) and (9) and re-decode it. If the re-decoding succeeds, then $\bigoplus_{\forall l \in \{i_1, \cdots, i_h\}} \mathbf{c}_l$ and $\bigoplus_{\forall l \in \{i_{h+1}, \cdots, i_{e_{\min}}\}} \mathbf{c}_l$ are known. By repeating the previous process, $\mathbf{c}_{i_1}, \cdots, \mathbf{c}_{i_g}, \cdots, \mathbf{c}_{i_h}$ can also be split into two groups as

$$\left(\bigoplus_{\forall l \in \{i_{g+1}, \cdots, i_h\}} \mathbf{c}_l\right) = \left(\bigoplus_{\forall l \in \{i_1, \cdots, i_g\}} \mathbf{c}_l\right) \oplus \left(\bigoplus_{\forall l \in H_E(j) \backslash \{i_1, \cdots, i_h\}} \mathbf{c}_l\right) \quad (10)$$

and the same process can be applied to decode $\bigoplus_{\forall l \in \{i_{g+1}, \cdots, i_h\}} \mathbf{c}_l$. If the re-decoding is unsuccessful, then different grouping of the unsuccessfully decoded codewords can be tried. This process is repeatedly performed until all errors are corrected.

The flowchart of the proposed decoding scheme is shown in Fig. 2, where only $e_{\min} \leqq 2$ is considered. This flowchart is based on the assumption that the vertical code is a linear systematic code. Note that four steps in the left bottom part of Fig. 2 are equivalent to the erasure decoding under binary erasure channel. It is well known that the *error correctability* τ of the vertical code is determined by the minimum Hamming distance $d_{\min}$ of the code, i.e., $\tau = d_{\min} - 1$ under binary erasure channel.

III. EXAMPLES

A. SPC Codes as Vertical Codes

Assume that an SPC code with length n is used as a vertical code of the product code and an LDPC code is used as a horizontal code. Then, the parity-check matrix H of the vertical code and its H_E can be written as

$$H = H_E = \begin{bmatrix} 1 & 1 & \cdots & 1 & 1 \end{bmatrix}.$$

From the parity-check matrix, it holds

$$\left(\bigoplus_{1 \leqq i \leqq n-1} \mathbf{s}_i\right) \oplus \mathbf{p}_1 = \mathbf{0}.$$

After decoding each of n horizontal codewords, there may exist e unsuccessfully decoded horizontal codewords. Since H, H_E, and H_P have only one row whose entries are all 1, $e_{\min}$ is always equal to e. If $e_{\min} = 1$ and the unsuccessfully decoded codeword is parity codeword $\mathbf{p}_1$, we already successfully decode the whole systematic horizontal codewords $\mathbf{s}_1 \sim \mathbf{s}_{n-1}$. If $e_{\min} = 1$ and the unsuccessfully decoded codeword is a systematic codeword $\mathbf{s}_i$, this codeword can be easily corrected by the even parity condition for the vertical code as

$$\mathbf{s}_i = \left(\bigoplus_{1 \leqq l \leqq n-1, l \neq i} \mathbf{s}_l\right) \oplus \mathbf{p}_1. \quad (11)$$

If $e_{\min} = 2$ and two systematic codewords $\mathbf{s}_{i_1}$ and $\mathbf{s}_{i_2}$ are unsuccessfully decoded, (11) can be rewritten as

$$\mathbf{s}_{i_1} = \left(\bigoplus_{1 \leqq l \leqq n-1, l \neq i_1} \mathbf{s}_l\right) \oplus \mathbf{p}_1 = \mathbf{s}_{i_2} \oplus \left(\bigoplus_{1 \leqq l \leqq n-1, l \neq i_1, i_2} \mathbf{s}_l\right) \oplus \mathbf{p}_1.$$

We can exactly calculate $\left(\bigoplus_{1 \leqq l \leqq n-1, l \neq i_1, i_2} \mathbf{s}_l\right) \oplus \mathbf{p}_1$ because all these codewords are successfully decoded. With this result and the soft-decision vector for $\mathbf{s}_{i_2}$, we have another independently received soft-decision vector $\mathbf{r}'_{i_1}$ for a codeword $\mathbf{s}_{i_1}$ as

$$\mathbf{r}'_{i_1} = \mathbf{r}_{i_2} \boxplus \left(\boxplus_{1 \leqq l \leqq n, l \neq i_1, i_2} \bar{\mathbf{r}}_l\right).$$

Similar to the type-I H-ARQ, we can add two soft-decision vectors $\mathbf{r}_{i_1}$ and $\mathbf{r}'_{i_1}$ and then re-decode it to obtain $\mathbf{s}_{i_1}$. If one of two unsuccessfully decoded codewords is $\mathbf{p}_1$, it is easy to show that the same procedure can be applied.

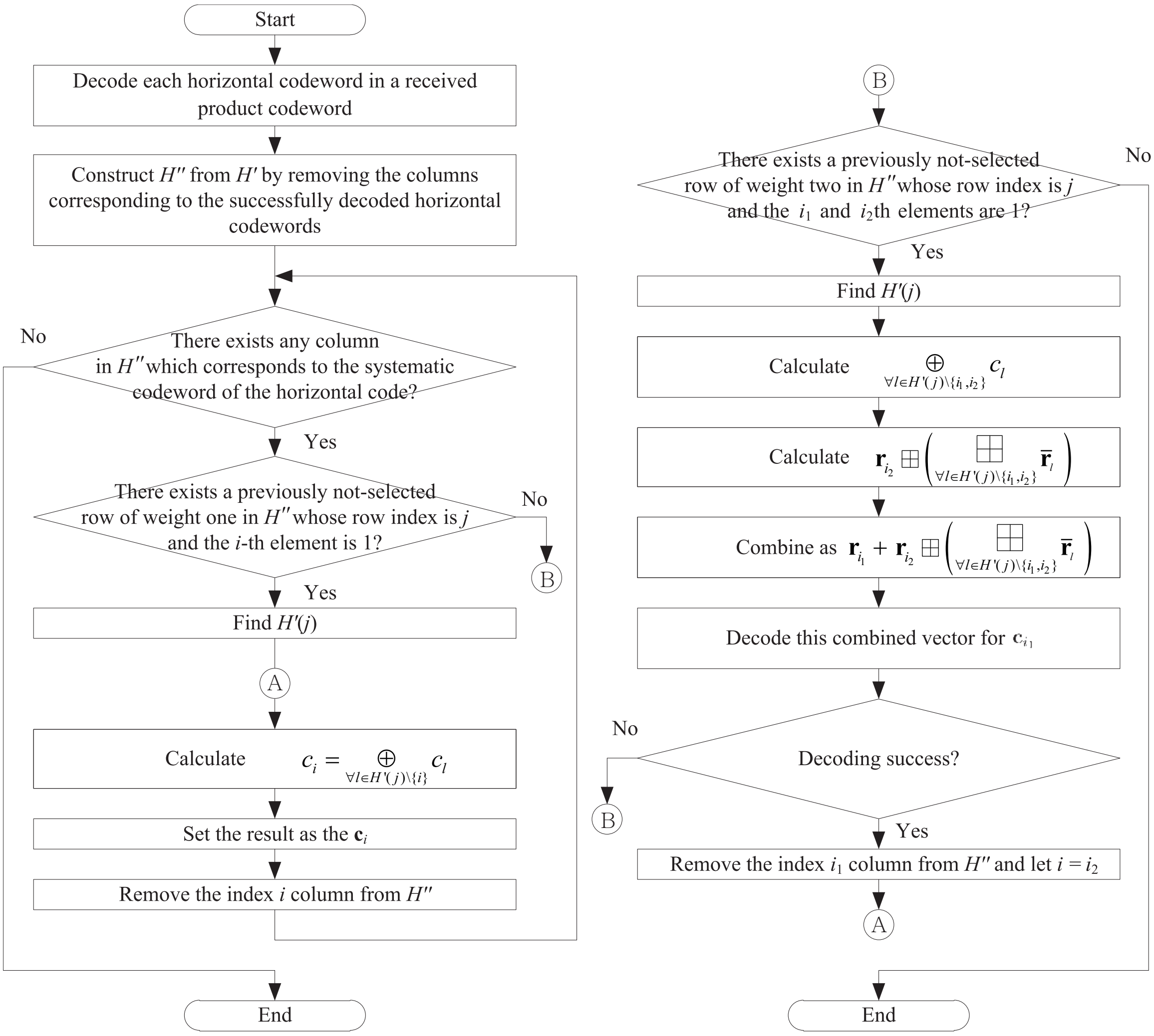


Fig. 2. Flowchart of the proposed decoding process.

If $e_{\min} = 3$ and three systematic codewords $\mathbf{s}_{i_1}$, $\mathbf{s}_{i_2}$, and $\mathbf{s}_{i_3}$ are unsuccessfully decoded, (11) can be rewritten as

$$\begin{aligned}\mathbf{s}_{i_1} &= \left(\bigoplus_{1\leqq l\leqq n-1, l\neq i_1} \mathbf{s}_l\right) \oplus \mathbf{p}_1 \\ &= \mathbf{s}_{i_2} \oplus \mathbf{s}_{i_3} \oplus \left(\bigoplus_{1\leqq l\leqq n-1, l\neq i_1, i_2, i_3} \mathbf{s}_l\right) \oplus \mathbf{p}_1.\end{aligned}$$

Then, we can combine the soft-decision vectors for $\mathbf{s}_{i_2}$ and $\mathbf{s}_{i_3}$ by (1) and exactly calculate $\left(\bigoplus_{1\leqq l\leqq n-1, l\neq i_1, i_2, i_3} \mathbf{s}_l\right)\oplus\mathbf{p}_1$. By using these vectors, we derive another independently received soft-decision vector $\mathbf{r'}_{i_1}$ for the codeword $\mathbf{s}_{i_1}$ as

$$\mathbf{r'}_{i_1} = \mathbf{r}_{i_2} \boxplus \mathbf{r}_{i_3} \boxplus \left(\boxplus_{1\leqq l\leqq n, l\neq i_1, i_2, i_3} \bar{\mathbf{r}}_l\right).$$

Finally, we add two soft-decision vectors $\mathbf{r}_{i_1}$ and $\mathbf{r'}_{i_1}$ and re-decode it for $\mathbf{s}_{i_1}$. If the re-decoding fails, then we can retry the same decoding procedure for $\mathbf{s}_{i_2}$ or $\mathbf{s}_{i_3}$ instead of $\mathbf{s}_{i_1}$. In the case of $e_{\min} \geqq 4$, the similar process can be applied.

B. $(7,4)$ Hamming Codes as Vertical Codes

Assume that a $(7,4)$ Hamming code is used as a vertical code of the product code and an LDPC code is used as a horizontal code. Then, the parity-check matrix H of the vertical code and its H_E can be written as

$$H = \begin{bmatrix} 1 & 0 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow H_E = \begin{bmatrix} 1 & 0 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 \end{bmatrix}.$$

Then, the rows of H_E give the following check equations for

7 horizontal codewords $\mathbf{s}_1$, $\mathbf{s}_2$, $\mathbf{s}_3$, $\mathbf{s}_4$, $\mathbf{p}_1$, $\mathbf{p}_2$, and $\mathbf{p}_3$;

$$\mathbf{s}_1 \oplus \mathbf{s}_3 \oplus \mathbf{s}_4 \oplus \mathbf{p}_1 = \mathbf{0} \quad (12)$$
$$\mathbf{s}_1 \oplus \mathbf{s}_2 \oplus \mathbf{s}_3 \oplus \mathbf{p}_2 = \mathbf{0} \quad (13)$$
$$\mathbf{s}_2 \oplus \mathbf{s}_3 \oplus \mathbf{s}_4 \oplus \mathbf{p}_3 = \mathbf{0} \quad (14)$$
$$\mathbf{s}_2 \oplus \mathbf{s}_4 \oplus \mathbf{p}_1 \oplus \mathbf{p}_2 = \mathbf{0} \quad (15)$$
$$\mathbf{s}_1 \oplus \mathbf{s}_2 \oplus \mathbf{p}_1 \oplus \mathbf{p}_3 = \mathbf{0} \quad (16)$$
$$\mathbf{s}_1 \oplus \mathbf{s}_4 \oplus \mathbf{p}_2 \oplus \mathbf{p}_3 = \mathbf{0} \quad (17)$$
$$\mathbf{s}_3 \oplus \mathbf{p}_1 \oplus \mathbf{p}_2 \oplus \mathbf{p}_3 = \mathbf{0}. \quad (18)$$

After decoding each of seven horizontal codewords, there may exist e horizontal codewords, $0 \leqq e \leqq 7$, which fail to be correctly decoded. If $e = 1$, at least one check equation from (12)$\sim$(18) contains that unsuccessfully decoded horizontal codeword, which can be easily corrected. It is easy to see that selecting proper check equation to correct an erroneous horizontal codeword is equivalent to finding a weight-1 row in H_P.

The same approach can be used to correct more than one error. For example, let $\mathbf{s}_1$, $\mathbf{s}_2$, and $\mathbf{s}_3$ be three unsuccessfully decoded horizontal codewords. Then, we have

$$H_P = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

and each recovery process for $\mathbf{s}_1$, $\mathbf{s}_2$, and $\mathbf{s}_3$ can be conducted using (15), (17), and (18), which correspond to the sixth, the fourth, and the last rows of H_P, respectively.

However, for three unsuccessfully decoded codewords $\mathbf{s}_1$, $\mathbf{s}_2$, and $\mathbf{s}_4$, there is no weight-1 row in the following H_P constructed by selecting the first, second, and fourth columns of H_E, which has $e_{\min} = 2$ as

$$H_P = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

Instead, as the Case 2) in the subsection II-B, by using (12) corresponding to the first row of the above H_P, we can modify the sign of each element of the soft-decision vector of $\mathbf{s}_4$ using the correctly decoded codewords $\mathbf{s}_3$ and $\mathbf{p}_1$, which results in another soft-decision vector for $\mathbf{s}_1$. This newly created soft-decision vector can be added to the original soft-decision vector for $\mathbf{s}_1$ and the re-decoding can be performed for this combined vector. If the re-decoding fails, we can try the same decoding procedure for another check equation corresponding to a weight-2 row in H_P.

Now, we consider more severe case, for example, there is only one successfully decoded horizontal codeword in a codeword matrix ($e = 6$), say $\mathbf{s}_3$. Using (12), we have the following re-decoding steps:

1. Combine the soft-decision vectors of $\mathbf{s}_4$ and $\mathbf{p}_1$ using the operation (1) to have the soft-decision vector for $\mathbf{s}_4 \oplus \mathbf{p}_1$;
2. Modify the sign of each element of the resulting soft-decision vector using $\mathbf{s}_3$ to obtain the soft-decision vector for $\mathbf{s}_3 \oplus \mathbf{s}_4 \oplus \mathbf{p}_1$.

The resulting soft-decision vector for $\mathbf{s}_3 \oplus \mathbf{s}_4 \oplus \mathbf{p}_1$ can be considered as the other independently received vector for $\mathbf{s}_1$. Thus this newly created soft-decision vector can be added to the original soft-decision vector for $\mathbf{s}_1$ and the re-decoding can be performed for this combined soft-decision vector. If the decoding is successful, then it gives $e = 5$ case. If the decoding is not successful, we can also try other check equation until the decoding is successful or there is no remaining check equation available.

IV. COMBINED-DECODABILITY OF THE PROPOSED DECODING SCHEME

In this section, the combined-decodability of the proposed decoding scheme is defined and the relation between the combined-decodability of the proposed decoding scheme and the vertical code with parity-check matrix H is investigated. Although the soft information combining of the proposed decoding scheme can be applied for any value of $e_{\min}$, the analysis of combined-decodability is performed by only considering the cases of $e_{\min} \leqq 2$ in this section.

A. Definitions and Properties of Combined-Decodability

Definition 1 (ϵ combinable) For a given H of a vertical code $\mathcal{C}$ and a fixed ϵ, if every possible $M \times \epsilon$ matrix H_P contains at least one row of Hamming weight one or two, the vertical code $\mathcal{C}$ is said to be ϵ *combinable*.

Definition 2 (η combined-decodable) If a vertical code $\mathcal{C}$ is ϵ combinable for all $\epsilon \leqq \eta$, then $\mathcal{C}$ is said to be η *combined-decodable*. If $\mathcal{C}$ is not $(\eta+1)$ combined-decodable but η combined-decodable, then η is called the combined-decodability of $\mathcal{C}$.

Using the above definitions, the length of vertical code can be bounded as in the following lemmas and theorems.

Lemma 3: If a vertical code $\mathcal{C}$ with a parity-check matrix H is ϵ combinable, any vertical code $\tilde{\mathcal{C}}$ whose parity-check matrix $\tilde{H}$ is constructed by selecting ϵ or more columns from H is also ϵ combinable.

Proof: A set of all possible $\tilde{H}''$'s which are constructed by selecting ϵ columns from $\tilde{H}'$ is contained in the set of all possible H_P's which are constructed by selecting ϵ columns from H_E. □

Lemma 4: All vertical codes $\mathcal{C}$ are 1 and 2 combinable.

Proof: Since there is no all-zero column in the parity-check matrix H of any vertical code $\mathcal{C}$, all vertical codes $\mathcal{C}$ are both 1 and 2 combinable. □

Lemma 5: If a vertical code $\mathcal{C}$ with an $m \times n$ parity-check matrix H is 3 combinable, then $n \leqq 2M$, where $M = 2^m - 1$. Moreover, when $n = 2M$, $\mathcal{C}$ is 3 combinable if and only if H is constructed by using every nonzero m-tuple column vector exactly twice as its columns.

Proof: If $n \geqq 2M+1$, there should exist three identical columns in H_E. If these three identical columns are selected to construct H_P, each row weight of H_P is either zero or three. Thus $\mathcal{C}$ cannot be 3 combinable.

(Only if part): For a parity-check matrix H of a 3 combinable code $\mathcal{C}$ with $n = 2M$, suppose that all the nonzero m-tuple column vectors do not appear exactly twice in H. Then, H_E should have at least three identical columns. If these three identical columns are selected to construct H_P, it contradicts the assumption that $\mathcal{C}$ is 3 combinable.

(If part): Clearly, $n = 2M$ because the number of nonzero m-tuple column vectors is $M = 2^m - 1$. If we construct H_P by choosing any three columns from H_E, there should be at least one distinct column from the others and therefore we can always find a row of weight one or two in H_P. □

From Lemmas 4 and 5, the following theorem can be stated without proof.

Theorem 6: The combined-decodability of SPC codes is two.

In order to find the combined-decodability of Hamming codes, the following lemmas are needed.

Lemma 7: If a vertical code $\mathcal{C}$ with an $m \times n$ parity-check matrix H is 4 combinable, then $n \leqq 3M$, where $M = 2^m - 1$. Moreover, when $n = 3M$, $\mathcal{C}$ is 4 combinable if and only if H is constructed by using every nonzero m-tuple column vector exactly three times as its columns.

Proof: If $n \geqq 3M+1$, there should exist four identical columns in H_E. If these four identical columns are selected to construct H_P, each row weight of H_P is either zero or four. Thus $\mathcal{C}$ cannot be 4 combinable.

(Only if part): For a parity-check matrix H of a 4 combinable code $\mathcal{C}$ with $n = 3M$, suppose that all the nonzero m-tuple column vectors do not appear exactly three times in H. Then H_E should have at least four identical columns. If those four identical columns are selected to construct H_P, there is no row of weight one or two in H_P. Thus it contradicts the assumption that $\mathcal{C}$ is 4 combinable.

(If part): Since each nonzero m-tuple column vector appears exactly three times in H, $n = 3M$. If we choose any four columns from H_E, there should be at least one distinct column from the others.

Suppose that there is neither a weight-1 row nor a weight-2 row in H_P for $\epsilon = 4$. Since both H_E and H_P are closed under the row-wise addition, H_P can be of two distinct types, that is, the first type has $\begin{bmatrix}1 & 1 & 1 & 1\end{bmatrix}$ as its nonzero rows and the second type has one 4-tuple of weight three, say $\begin{bmatrix}1 & 1 & 1 & 0\end{bmatrix}$, as its nonzero rows such that

$$H_P = \begin{bmatrix} 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ \vdots & \vdots & \vdots & \vdots \\ 1 & 1 & 1 & 1 \end{bmatrix} \text{ or } \begin{bmatrix} 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 1 & 1 & 1 & 0 \end{bmatrix}.$$

However, note that the latter matrix cannot be obtained because there is no all-zero column in H_P. Therefore the former matrix is the only possible matrix for H_P to have rows of weight zero or four.

Based on this result, we can see that there should be a row of weight one or two in H_P. □

The following corollary is given without proof.

Corollary 8: If a vertical code $\mathcal{C}$ is 3 combinable, then $\mathcal{C}$ is also 4 combinable and thus $\mathcal{C}$ is 4 combined-decodable.

Lemma 9: Suppose that vertical code $\mathcal{C}$ has minimum distance $d_{\min}$ which is larger than or equal to 3. If $\mathcal{C}$ with an $m \times n$ parity-check matrix H is 5 combinable, then $n \leqq M$, where $M = 2^m - 1$. Moreover, when $n = M$, $\mathcal{C}$ is 5 combinable if and only if H is constructed by using every nonzero m-tuple column vector only once as its columns.

Proof: Since $d_{\min}$ is larger than or equal to 3, H should not contain identical columns. Therefore, $n \leqq M$ is trivial and the first part is proved.

(Only if part): For $n = M$ and $d_{\min} \geqq 3$, H is uniquely constructed by using every nonzero m-tuple column vector only once as its columns.

(If part): Suppose that there is neither a weight-1 row nor a weight-2 row in H_P for $\epsilon = 5$. Since both H_E and H_P are closed under the row-wise addition, there are only two types of H_P such as

1. H_P should contain $\begin{bmatrix}1 & 1 & 1 & 1 & 1\end{bmatrix}$ as its only nonzero rows;
2. H_P should contain $\begin{bmatrix}1 & 1 & 1 & 1 & 0\end{bmatrix}$, $\begin{bmatrix}1 & 1 & 0 & 0 & 1\end{bmatrix}$, and $\begin{bmatrix}0 & 0 & 1 & 1 & 1\end{bmatrix}$ at least once as its nonzero rows.

Note that the second type represents all the cases of three nonzero rows having 1, 2, and 2 zeros in distinct positions, respectively. Then we have

$$H_P = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix} \text{ or } \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix}.$$

Therefore, in order for $\mathcal{C}$ to be 5 combinable, H should contain neither five identical columns nor two distinct pairs of columns.

If H is constructed by using every nonzero m-tuple column vector only once as its columns, it is clear that H has neither five identical columns nor two distinct pairs of columns and $\mathcal{C}$ is 5 combinable. Thus the proof is done. □

For 5 combinable vertical code $\mathcal{C}$ without restriction on the minimum distance, it is not difficult to verify that the upper bound in Lemma 9 becomes $n \leqq 2M + 2$. This bound can be proved by similar method used in the proof of Lemma 9.

Table 1. Distribution of H_P for all possible horizontal codeword errors in the proposed decoding scheme with Hamming codes.

(n,k)	e	Number of distinct H_P for e		
		Total $\binom{n}{e}$	With rows of Hamming weight one or two	Without rows of Hamming weight one or two
$(7,4)$	6	7	0	7
	5	21	21	0
	4	35	35	0
	3	35	35	0
	2	21	21	0
$(15,11)$	10	3,003	0	3,003
	9	5,005	396	4,609
	8	6,435	2,772	3,663
	7	6,435	5,313	1,122
	6	5,005	4,900	105
	5	3,003	3,003	0
	4	1,365	1,365	0
	3	455	455	0
$(31,26)$	10	44,352,165	20,827,060	23,525,105
	9	20,160,075	15,860,120	4,299,955
	8	7,888,725	7,681,550	207,175
	7	2,629,575	2,597,965	31,610
	6	736,281	735,196	1,085
	5	169,911	169,911	0
	4	31,465	31,465	0
	3	4,495	4,495	0
$(63,57)$	7	553,270,671	552,631,671	639,000
	6	67,945,521	67,935,755	9,766
	5	7,028,847	7,028,847	0
	4	595,665	595,665	0
	3	39,711	39,711	0

B. Combined-Decodability of Hamming Codes

In this subsection, we will derive the combined-decodability of Hamming codes.

Lemma 10: Hamming codes as vertical codes are neither 6 combinable nor 7 combinable.

Proof: For a $(7,4)$ Hamming code, H_E is given in subsection III-B. The first three rows in H_E show all possible nonzero column vectors of length 3 and the rows in H_E are all possible linear combinations of the first three rows except all-zero row. Since all rows in H_P have weight four, a $(7,4)$ Hamming code is not 7 combinable.

For a $(2^m-1, 2^m-m-1)$ Hamming code, the parity-check matrix consists of all nonzero m-tuple column vectors. Using only column permutation, we can reorder the parity-check matrix as

$$H = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \cdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 1 & 0 & 1 & 1 & 1 & 0 & 0 & \cdots \\ 0 & 1 & 0 & 1 & 1 & 1 & 0 & \cdots \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 & \cdots \end{bmatrix}.$$

Note that the first seven column vectors have zeros in the first $m-3$ positions and they should be all different from each other. Since there is no all-zero column vector in H, there should appear all nonzero 3-tuple column vectors in the last three rows corresponding to the first seven columns. Then, we can construct H_E by using all possible non-trivial linear combinations of rows in H, and H_P by selecting the first seven columns from H_E. It is clear that such H_P is made up of all-zero rows and the rows of H_E for a $(7,4)$ Hamming code.

Since the weight of rows of H_P is either 0 or 4, Hamming codes are not 7 combinable. It is also easy to check that Hamming codes are not 6 combinable because we can always find an H_P for $e=6$ which consists of weight-0, weight-3, and weight-4 rows by choosing any 6 columns from the above H_P for $e=7$. □

Theorem 11: The combined-decodability of Hamming codes as vertical codes is five.

Proof: From Lemmas 3, 4, 5, 7, and 9, Hamming codes are 5 combined-decodable. The theorem is directly proved by this result and Lemma 10. □

Table 1 shows the distribution of H_P for all possible unsuccessfully decoded horizontal codewords when Hamming codes are used as vertical codes. It clearly shows that combined-decodability of any Hamming code is five. Table 2 shows the distribution of H_P for all possible unsuccessfully decoded horizontal codewords when double parity-check (DPC) codes are used as vertical codes. It shows that the combined-decodabilities of $(6,4)$ and $(12,10)$ DPC codes are four and two, respectively.

Table 2. Distribution of H_P for all possible horizontal codeword errors in the proposed decoding scheme with DPC codes.

(n,k)	e	Number of distinct H_P for e		
		Total $\binom{n}{e}$	With rows of Hamming weight one or two	Without rows of Hamming weight one or two
$(6,4)$	5	6	0	6
	4	15	15	0
	3	20	20	0
	2	15	15	0
$(12,10)$	7	792	0	792
	6	924	84	840
	5	792	360	432
	4	495	492	3
	3	220	208	12
	2	66	66	0

Table 3. Candidates for the vertical codes in the proposed decoding scheme.

(n,k)	Code	Generator polynomial	$d_{\min}$	τ correctability	η combined-decodability
$(63,57)$	Hamming	x^6+x+1	3	2	5
$(31,26)$		x^5+x^2+1			
$(15,11)$		x^4+x+1			
$(7,4)$		x^3+x+1			
$(12,10)$	DPC	x^2+x+1	2	1	2
$(6,4)$					4
$(k+1,k)$	SPC	$x+1$			2

To analyze the combined-decodability of the proposed decoding scheme, some simple linear codes such as SPC codes, DPC codes, and Hamming codes are considered as vertical codes, the parameters of which are listed in Table 3.

V. SIMULATION RESULTS

In this section, numerical analysis of the proposed decoding scheme is performed. The performance of LDPC codes with $R=1/2$ and $R=5/6$ in two different lengths $n=2,304$ and $n=1,152$ in IEEE 802.16e standard [22] are compared with that of the proposed decoding scheme. The maximum number of iterations for iterative belief propagation decoding is set to 50 for the entire simulations.

For the proposed decoding scheme, the $(24,23)$ SPC code is selected as the vertical code. To maintain the same overall code rate, the LDPC codes obtained by puncturing the above codes in IEEE 802.16e standard are used as the horizontal codes. By using the puncturing rate $1/24$, the overall code rates for the proposed decoding scheme are also set to $1/2$ and $5/6$.

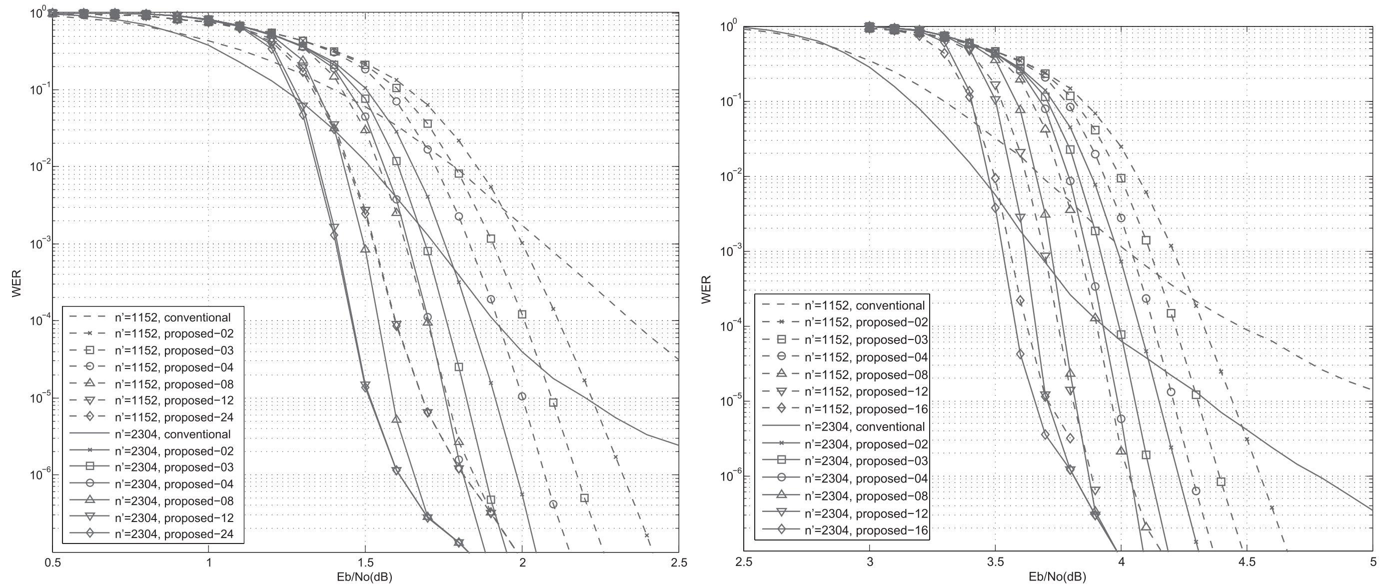
Fig. 3 shows that the proposed decoding scheme with $(24,23)$ SPC vertical code outperforms the conventional LDPC codes in IEEE802.16e standard for $R=1/2$ and $R=5/6$, especially in high SNR region, where 'Proposed-e' means that up to e horizontal codeword errors are combined-decoded. Error correcting performance of each decoding scheme is shown in Fig. 3 by word error rate (WER) of LDPC codewords for the conventional scheme and horizontal LDPC codewords for the proposed scheme. Even though the proposed decoding scheme does not correct all error patterns of e or less horizontal codeword errors, Fig. 3 shows remarkable performance improvement. The simulation results also show that both the waterfall and error-floor performance of the LDPC code can be improved by constructing a product code and performing the proposed decoding scheme.

Additional computational complexity for the proposed decoding scheme may be significant for low SNR region. However, for high SNR region, decoding failure of more than one horizontal codeword in a codeword matrix is rarely occurred. Thus, in spite of remarkable performance improvement, additional computational complexity per codeword becomes negligible for high SNR region.

We showed the numerical results only for the case of SPC codes being used as vertical codes. Also, some of possible grouping methods for the erroneous horizontal codewords, i.e., grouping into one codeword and others, are considered for the case of $e_{\min} \geqq 3$. If we use more powerful codes such as Hamming codes as vertical codes or consider more various grouping methods for the erroneous horizontal codewords, the performance of the proposed decoding scheme is believed to be improved but their decoding complexity increases.

VI. CONCLUSION

The proposed decoding scheme adopts a simple product code structure to have a strong post-processing for good error correcting performance of LDPC codes. It mainly takes advantage of combining two independently received soft-decision data for the same codeword. The proposed decoding scheme can achieve very low error rate, which is very difficult only with a single LDPC code and the conventional LDPC decoding. Especially,



(a) (b)

Fig. 3. Performance comparison of the conventional decoding of LDPC codes in IEEE802.16e and the proposed decoding with $(24, 23)$ SPC code as a vertical code: (a) $R = 1/2$ and (b) $R = 5/6$.

the proposed decoding scheme shows better error correcting capability in high SNR region. Also, the decoding capability of the proposed decoding scheme, which is described as *combined-decodability*, is exactly analyzed using parity-check matrices of vertical codes for which SPC codes and Hamming codes are used.

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